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Theory of Low Energy Nucleon-Nucleon Scattering.

II. High Waves.

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Summary. — The theory developed in a previous paper (1) is here applied to the special case of high waves. The distinction between « high » and « low » waves is discussed in the text, and it is hoped that the formulae given in this paper will yield a valid representation of the nucleon-nucleon partial wave amplitudes for values of the orbital momentum $l \geqslant 2$. For these high waves the cut in the energy plane due to three pion or more massive intermediate states is supposed to give a negligible contribution. In this case the representation given in I reduces to a set of explicit formulae for the partial wave amplitudes. These formulae contain explicitly the effect of the Born term and rescattering corrections in pion nucleon scattering as well as the effect of a possible pion-pion interaction.

1. – Introduction.

In a previous paper (1) a theory of low energy nucleon-nucleon scattering was developed from the viewpoint of the double dispersion relation technique. In this paper we shall apply the results obtained to a particularly simple case, *i.e.* that of higher waves. In order to clarify better this statement let us recall briefly some results of I. Starting from a Mandelstam representation for a particular set of functions $c_j^T(wt\bar{t})$, we obtained for another set of functions

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(1) D. AMATI, E. LEADER and B. VITALE: *Nuovo Cimento*, **17**, 68 (1960). We refer to this paper (called I in the following) for notation, symbols and the general ideas of the approach.

$p_j^T(wt\bar{t})$, related to the previous ones through (*)

$$c_j^T(wt\bar{t}) = \frac{D_j^T(t-\bar{t})}{w_D - w} + \left\{\sum_k U_{jk}(wt)\, p_k^T(wt\bar{t})\right\} + (-1)^{j+T}\{t \to \bar{t}\} \tag{1.1}$$

a dispersion relation of the type (**)

$$p_j^{\pm}(wt\bar{t}) = \begin{pmatrix} 0 \\ \\ -\frac{1}{2}\frac{g^2\delta_{j5}}{\mu^2 - t} \end{pmatrix} + \left\{\frac{1}{\pi}\int_{4\mu^2} \frac{\varrho_j^{\pm}(wt')\,\mathrm{d}t'}{t'-t} + \frac{1}{\pi}\int_{4m^2} \frac{\chi_j^{\pm}(xt)\,\mathrm{d}x}{x-w}\right\} \mp (-1)^j\{w \to \bar{t}\}, \tag{1.2}$$

where the function $\chi_j(x, t)$—coming from 3 pion or more massive intermediate states in $\mathcal{N}\bar{\mathcal{N}} \to \mathcal{N}\bar{\mathcal{N}}$—had singularities in t starting only at $t = 9\mu^2$.

The approach used in the whole calculation was to single out the contribution of all the lower lying singularities in a particular variable and approximate the contributions of the higher cuts by slowly varying functions on the variables in question, *i.e.* the lower terms in a polynomial expansion. It is difficult to say *a priori* how many terms must be retained for any of these expansions: the theory, however, makes sense only if this number is reasonably small. The final answer to such a question clearly must come from the comparison of the results with experiment. The fact that $\chi_j(xt)$ has cuts in t starting only at $9\mu^2$ allows us to suppose within the framework of our theory, that the last integral of (1.1) will contribute a dependence only on the lowest powers in t. This means that if we project eq. (1.1) over high powers of t, that term would have to give a very small contribution. It is easy to understand that such a requirement means that the high waves are essentially given by the pole terms ([2]) and the two pion contribution that gives rise to the integration over ϱ in (1.1) ([3]).

(*) The matrix $U_{jk}(wt)$ which is linear in t is given explicitly in Section **3** of this paper.

(**) Recall

$$p_i^1 = 3p_i^+ + 2p_i^- , \qquad p_i^0 = 3p_i^+ - 6p_i^- .$$

([2]) The pole term has been treated explicitly by P. Cziffra, M. H. MacGregor, M. J. Moravcsik and H. P. Stapp: *Phys. Rev.*, **114**, 880 (1959). Their M_{ij} coincide with the pole contribution for T_{ij} for waves with $l \geqslant 2$. The difference between our pole contribution and their OPEC term for s and p waves (absolutely unimportant for the present purpose) comes from the fact that they defined OPEC as the Born term while our pole term contains the residue of the Born term at the pole. The numerical analysis for p-p scattering at 310 MeV carried out by M. H. MacGregor, M. Moravcsik and H. P. Stapp, *UCRL*-5566, shows that the waves with $l \geqslant 6$ are well explained by the pole term.

([3]) An approach to the high waves, in some sense analogous to ours, has been recently carried out by A. D. Galanin, A. F. Grashim, B. L. Joffe anz I. Ya. Po-

This argument can also be stated in another way. The term χ is essentially there in order to satisfy unitarity in $\mathcal{N}$-$\mathcal{N}$ scattering. In fact it is just by imposing the unitarity condition that (1.1) can be converted into an integral equation whose solution determines χ. To neglect χ is justified for those waves which are sufficiently small so that unitarity gives a non-restrictive condition (*).

In our case, even neglecting χ, the amplitudes will have an imaginary part coming from ϱ. In fact as is clearly shown in I, ϱ has a cut in w starting at $(2m)^2$ that, however, for the region of energy we are interested in, comes only from the fourth order perturbative contribution to the ϱ. So we can define « high waves » to be those for which the imaginary part of the partial amplitudes are essentially given by the 4-th order calculation (explicitely given in Section **3**·1*b*). Let us discuss the form of ϱ as given in (3.43) of I a little further. We see that it has essentially 3 contributions: the integrals over the functions $\varkappa$, the terms containing $\alpha(t)$, $\beta(t)$, and those with $\lambda(t)$, $\eta(t)$.

The first contribution, as discussed in I, comes from the 4-th order perturbative graphs and the π-$\mathcal{N}$ rescattering corrections to it. It brings all the information on π-$\mathcal{N}$ scattering into $\mathcal{N}$-$\mathcal{N}$ scattering. The second contribution is the projection of the first over the 2π intermediate states in s and p waves and the last contribution comes from the lowest waves of the $\mathcal{N}\overline{\mathcal{N}} \to 2\pi$ process, for which up to now there is no much information.

In the next section we sketch the general method of reducing formulae (1.1) and (1.2) for the case of high waves. The relationship with the singlet and triplet partial wave amplitude is also derived.

In Section **3** we present the explicit formulae for the contributions to the amplitude arising from the 4-th order and π-$\mathcal{N}$ rescattering corrections as well as their s and p wave projections in the $\mathcal{N}\overline{\mathcal{N}}$ channel. The reader who is interested only in the general outline of the method can safely skip this arid and cumbersome section.

Section **4** discusses the contributions arising from low 2π waves in the $\mathcal{N}\overline{\mathcal{N}} \to 2\pi$ process. The relationship with π-π scattering through the unitarity

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MERANCHUK: *Žurn. Éxp. Teor. Fiz.*, **37**, 1663 (1959) and **38**, 475 (1960). They made, however, an assumption that simplifies radically the calculation but that can be not valid in the presence of a π-π resonance. This assumption is to take for the $\mathcal{N}\overline{\mathcal{N}} \to 2\pi$ amplitude its threshold ($t=4\mu^2$) behaviour (and eventually a first term in its expansion). It is clear that a possible bump due to π-π resonance would be badly approximated by such a form. In other words, such an approximation would give correctly only the very low part of the cut in t at $4\mu^2$, so that it would be meaningful only for extremely high waves for which, however, the pole contribution should in any case be by far the most important ([2]).

(*) A large *real* amplitude M would violate strongly the unitarity condition $M=\exp[i\delta]\sin\delta$ while a small one (for which $\sin^2\delta$ is vanishingly small) satisfies it approximately.

condition is shown. The effect of a strong, $T=1$, $J=1$ π-π resonance, suggested by nucleon electromagnetic structure and pion-nucleon scattering data ([4,5]) is then calculated explicitly in terms of the parameters characterising this resonance.

The formulae given may be used directly to predict the partial wave amplitudes in terms of the few parameters of the theory. Numerical calculations and comparison with experimental data are at present in progress.

2. – Reduction for high waves.

2·1. *General form of the reduction.* – We wish now to consider only those terms which will contribute to the high waves in the $\mathcal{N}\mathcal{N} \to \mathcal{N}\mathcal{N}$ channel, *i.e.* those terms in (1.2) whose dependence on t or $\bar{t}$ is strong. This means cuts in those variables starting in the region between $4\mu^2$ and $9\mu^2$. As mentioned in the introduction, we shall neglect the contribution of χ in (1.2) for the high waves. The function $\varrho_j(wt)$, coming from the two pion contribution, had the form

$$\varrho_j(wt) = \frac{1}{\pi}\int_{4m^2} \frac{\varkappa_j(xt)\,\mathrm{d}x}{x-w} + [s,\ p \text{ terms}]\,. \tag{2.1}$$

The form of the terms s, p was given in I and consisted of the entire contributions to ϱ_j with s and p wave projection in the $\mathcal{N}\overline{\mathcal{N}}$ channel less the s and p wave parts of the first term in (2.1).

Further the term in $p_j^T(wt\bar{t})$ arising from the crossing $w \to \bar{t}$ is of the type

$$\int_{4\mu^2} \frac{\varrho_j(\bar{t}t')\,\mathrm{d}t'}{t'-t} = \int_{4\mu^2}\int_{4m^2} \frac{\varkappa_j(t'x)}{(t'-t)(x-\bar{t})}\,\mathrm{d}t'\,\mathrm{d}x\,, \tag{2.2}$$

so that using the relation

$$\frac{1}{(t'-t)(x-\bar{t})} = \left[\frac{1}{t'-t} + \frac{1}{x-\bar{t}}\right]\frac{1}{x+t'-4m^2+w} \tag{2.3}$$

one sees that one may retain just the first term, since the second, being integrated from $4m^2$, is weakly dependent on $\bar{t}$ in the region of interest ($\bar{t}$ negative).

([4]) W. FRAZER and J. FULCO: *Phys. Rev. Lett.*, **2**, 365 (1959); *Phys. Rev.*, **117**, 1609 (1960).

([5]) J. BOWCOCK, W. N. COTTINGHAM and D. LURIÉ: *Nuovo Cimento*, **16**, 918 (1960). We refer to this paper (called BCL in the following) for the meaning of some symbols used in Section **4**. W. FRAZER and F. FULCO: *Phys. Rev.*, **119**, 1420 (1960).

Finally then, keeping only the terms which contribute to the high waves, we have:

$$(2.4)\qquad p_j^{\pm}(wt\bar{t}) = \begin{pmatrix} 0 \\ -\frac{1}{2}\frac{g^2\delta_{j5}}{\mu^2 - t} \end{pmatrix} + \left\{\frac{1}{\pi}\int_{4\mu^2} \frac{\varrho_j^{\pm}(wt')\,\mathrm{d}t'}{t' - t}\right\} \mp (-1)^j\{w \to \bar{t}'\}\,,$$

where $\bar{t}' = t' + w - 4m^2$.

It is easy to see from the form of the ϱ_j (eq. (2.1)) that the $p_j(wt\bar{t})$ develop imaginary parts in the $\mathcal{N}\mathcal{N}$ channel only when the denominator $x - w$ vanishes. *I.e.* if the $\varkappa_j(xt)$ are non-zero at $x = w$ then the imaginary part of the $p_j(wt)$ is exactly equal to $\varkappa_j(wt)$. The real part, on the other hand, is given by the principal value integral of (2.1).

2·2. *General form of the weight functions.* – We had seen in I that the $\varkappa(xt)$ functions were given by double integrals of the form

$$(2.5)\qquad \varkappa(xt) = \iint \mathrm{d}s'\,\mathrm{d}s''\,F(s's''tx)\,\bar{\sigma}_{A,B}(s''t)\,\bar{\sigma}_{A,B}(s't)\,,$$

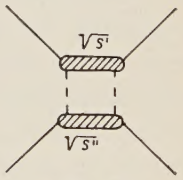


Fig. 1.

where $F(s's''tx)$ was a known function and the $\bar{\sigma}_{A,B}(st)$ were the spectral functions occurring in π-nucleon scattering.

The integrations over $\mathrm{d}s'$ and $\mathrm{d}s''$ represented a sum over all the possible mass states $(\sqrt{s'},\ \sqrt{s''})$ in which two pions are exchanged (see Fig. 1) and the range of integration was bounded to be below the curve

$$(2.6)\qquad s's'' + (s' + s'')\left[\frac{t}{2} - \mu^2 - m^2\right] + (m^2 - \mu^2)^2 - \frac{x}{2}(t - 4\mu^2) +$$

$$+ \sqrt{[s'^2 + s'(t - 2\mu^2 - 2m^2) + (m^2 - \mu^2)^2][s''^2 + s''(t - 2\mu^2 - 2m^2) + (m^2 - \mu^2)^2]} = 0\,.$$

As we have already said in I, the spectral functions $\bar{\sigma}_{A,B}$ have been given by CHEW, GOLDBERGER, LOW and NAMBU (6) taking into account in the π-$\mathcal{N}$ re-scattering corrections only the 33 resonance as:

$$\bar{\sigma}_B^{\pm} = \pi g^2\,\delta(s - m^2) + \sigma_B^{\pm}\,,$$

$$\bar{\sigma}_A^{\pm} = \sigma_A^{\pm}\,,$$

(6) G. F. CHEW, M. L. GOLDBERGER, F. LOW and Y. NAMBU: *Phys. Rev.*, **106**, 1337 (1957).

where

$$(2.7)\quad \begin{cases} \sigma_B^+(st) = -2\sigma_B^-(st) = \dfrac{8\pi}{3}\left[\dfrac{3}{E_{\pi\mathcal{N}}+m}\left(1+\dfrac{t}{2k_{\pi\mathcal{N}}^2}\right)-\dfrac{1}{E_{\pi\mathcal{N}}-m}\right]\dfrac{\sin^2\delta_{33}(s)}{k_{\pi\mathcal{N}}}, \\ \sigma_A^+(st) = -2\sigma_A^-(st) = \dfrac{8\pi}{3}\left[\dfrac{m+\sqrt{s}}{E_{\pi\mathcal{N}}+m}\,3\left(1+\dfrac{t}{2k_{\pi\mathcal{N}}^2}\right)+\dfrac{\sqrt{s}-m}{E_{\pi\mathcal{N}}-m}\right]\dfrac{\sin^2\delta_{33}(s)}{k_{\pi\mathcal{N}}}, \end{cases}$$

for $s \geqslant (m+\mu)^2$, and $\sigma_A^{\pm} = \sigma_B^{\pm} = 0$ for $s < (m+\mu)^2$;

$$k_{\pi\mathcal{N}}^2 = \frac{\frac{1}{4}(s-m^2-\mu^2)^2-\mu^2m^2}{s} \text{ and } E_{\pi\mathcal{N}} = \frac{s+m^2-\mu^2}{2\sqrt{s}}.$$

We require, however, to use formulae (2.7) in the region where $4\mu^2 \leqslant t \leqslant 9\mu^2$ which corresponds to an unphysical region in π-nucleon scattering. *I.e.* the momentum transfer is time-like and $\cos\theta$ takes values larger than one. Thus we must make an analytic continuation in t from the physical region, $t < 0$, up to the region of interest. This is done, as usual, by the continuation of the Legendre polynomial expansions. Since σ_A and σ_B have singularities in t beginning only at $16\mu^2$ (because the two pion contribution was explicitly separated out from the rest of the pion-nucleon scattering amplitude), we expect this expansion to be quite justified. However, even if the Legendre expansion does converge, the keeping of only the 33 wave could lead to difficulties, since a small d wave multiplied by $P_2(\cos\theta)$ for $\cos\theta > 1$ might induce a sizeable contribution in $\sigma_{A,B}$. Nevertheless, the extreme smallness of the d waves compared with the $p_{\frac{3}{2}}$ wave indicates that even in this region of $\cos\theta$, the retention of only δ_{33} should prove reasonable, and we believe that the use of eq. (2.7) is fairly justified.

Equations (2.4) and (2.1) now indicate that besides the one pion pole, there are two contributions to the amplitudes, the first arising from the iteration of π-nucleon scattering and the second consisting of all terms in which two pions are exchanged in either s or p states in the $\mathcal{N}\bar{\mathcal{N}} \to \mathcal{N}\bar{\mathcal{N}}$ channel. This latter term is therefore the difference between the s and p wave contributions of the whole $\mathcal{N}\bar{\mathcal{N}} \to 2\pi$ process and that fraction of it arising from the s and p wave projections of the first term.

The reason for this splitting, as will become clear in Section **4**, lies in the possibility of using unitarity for the whole 2π, s and p wave $\mathcal{N}\bar{\mathcal{N}} \to 2\pi$ and relating it to the pion-pion interaction. Even if there is no direct information on π-π interaction, several different phenomena depend on it indirectly and, therefore, on the parameters characterising such an interaction.

It will be simpler, therefore, in the following, to keep separate the s, p terms from the rest of the expressions for the $p_j^{\tau}(wt\bar{t})$.

Thus in Section **3** we shall give the explicit formulae obtained from the $\sigma_{A,B}$ contribution to ϱ_j and then in Section **4** discuss the contribution of the π-π interaction.

2·3. *Projection of partial wave amplitudes.* – We wish now to deduce expressions for the partial wave singlet and triplet amplitudes $T_{ij}(l)$ defined for istance in GOLDBERGER, NAMBU and OEHME (7). These are given in terms of the total amplitudes $T_{ij}(w, \cos\theta)$ by

$$T_{ij}(l) = \frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\int_{-1}^{+1} T_{ij}(\cos\theta)\, P_l^m(\cos\theta)\mathrm{d}\cos\theta\,, \tag{2.8}$$

where $m = |i-j|$, and the $T_{ij}(\cos\theta)$ are related to the amplitudes $c_j^r(wt\bar{t})$ by

$$T_{ij}(w, \cos\theta) = \sum_\alpha V_{ij}^\alpha(w, \cos\theta)\, c_\alpha^r(wt\bar{t})\,. \tag{2.9}$$

The matrix $V_{ij}^\alpha(w, \cos\theta)$ is given in the next section, but for the moment it is important to remember only that it is at most linear in $\cos\theta$.

The $\cos\theta$ integration of (2.8) is now carried out as follows. Firstly in (2.4) we rewrite the denominator $t'-t$ as

$$t' + 2k^2 - 2k^2\cos\theta$$

and then, changing the integration variable to

$$y = \frac{t'+2k^2}{2k^2}\,, \tag{2.9a}$$

we have (omitting for the moment the one pion pole term which is trivially handled)

$$p_j^\pm(wt\bar{t}) \sim \frac{1}{\pi}\int_{1+\frac{2\mu^2}{k^2}}^{1+\frac{t_{\max}}{2k^2}} \frac{\varrho_j^\pm(wy)\,\mathrm{d}y}{y-\cos\theta} \mp (-1)^j\,[w \to \bar{t}']\,, \tag{2.10}$$

where $t_{\max}$ is of the order of $9\mu^2$ (*). We then note that since the coefficient functions $V_{ij}^\alpha U_{\alpha\beta}$ relating the p_β to the T_{ij} are only bilinear in t (and there-

(*) $t_{\max}$ should strictly speaking be taken as $9\mu^2$, since we have previously considered cuts in t for $t > 9\mu^2$ to give a negligible contribution to the high waves. Since, however, the contribution of the 3 pion processes begins at $t=9\mu^2$ with zero slope, we expect that even for t' greater than $9\mu^2$, the 2 pion contribution will yield the dominant contribution to the cut. Thus we feel that the upper limit of the integration should, more accurately, be taken higher than $9\mu^2$, *i.e.* up to that value at which the 3 pion contributions begin to become important.

(7) M. L. GOLDBERGER, I. NAMBU and R. OEHME: *Ann. Phys.*, **2**, 226 (1957).

fore $\cos\theta$) we may replace $\cos\theta$ by y wherever it occurs in them and shift the entire expression under the y integration [since the difference between this and the original formulae is made out only by terms contributing to $l<2$]. The only $\cos\theta$ dependence left is then the denominator $y-\cos\theta$ of eq. (2.10) so that the integration can be carried out, yielding

$$T^{T}_{ij}(l)=\nu(Tmls)(2l+1)\frac{(l-m)!}{(l+m)!}\sum_{\alpha}\int_{1}^{1+\frac{t_{\max}}{2k^2}}V^{\alpha}_{ij}(w,y)\bar{c}^{T}_{\alpha}(w,y)(y^2-1)^{\frac{m}{2}}Q^{m}_{l}(y)\,dy\,, \tag{2.11}$$

where now

$$\bar{c}^{T}_{\alpha}(w,y)=\sum_{\beta}U_{\alpha\beta}(wy)\bar{p}^{T}_{\beta}(wy) \tag{2.12}$$

and

$$\bar{p}^{\pm}_{\beta}(wy)=\frac{\Theta(y-1-(2\mu^2/k^2))}{\pi}\{\varrho^{\pm}_{\beta}(w,y)\mp(-1)^{\beta}\varrho_{\beta}(\bar{t}',y)\}+ \tag{2.13}$$

$$+\begin{pmatrix}0\\ -\dfrac{g^2}{4k^2}\,\delta_{\beta 5}\,\delta\left(y-1-\dfrac{\mu^2}{2k^2}\right)\end{pmatrix}.$$

where Θ is the usual step function.

Note that in (2.12) we have no longer written the crossed term since it has been explicitly taken into account by the function $\nu(T,m,l,s)=1-(-1)^{l+m+T+s}$ in (2.11) where s is the total spin (*i.e.* 0 for singlet, 1 for triplet) of the $\mathcal{N}\mathcal{N}$ system.

Thus we are left finally with expressions for the $T_{ij}(l)$ involving only the y integration. This has to be carried out numerically and the details and explicit formulae appear in the next section.

3. – Explicit formulae.

We evaluate now explicitly the functions $\varrho^{\pm}_{j}(wy)$ whose real and imaginarys parts are to be used in (3.11), (12) and (13) to construct the $T_{ij}(l)$. It will be convenient to split up the contributions to the ϱ_j according to which terms in the $\bar{\sigma}_{A,B}$ give rise to them and to leave for Section **4** the contributions from the π-π interactions.

For simplicity, we shall replace the s dependence of $\sin^2\delta_{33}(s)$ of (2.9) by a zero-width resonance approximation. It is well known that the rescattering corrections in π-$\mathcal{N}$ scattering obtained by computing the dispersive integrals are rather independent of the assumed shape of the 3.3 resonance, so that a zero-width resonance form reproduces reasonably well the characteristics of the dispersive integral calculated from a more realistic phenomenological form

for the resonance, *e.g.* a Breit-Wigner formula. Normalizing the zero-width resonance for $\sin^2 \delta_{33}(s)$ so that the integral over it is equal to the area under the curve obtained from the usual effective range formula [4], we obtain

$$\sin^2 \delta_{33}(s) \sim \frac{M k_r^3}{6m^2} g^2 \delta(s - M^2) , \tag{3.1}$$

where M is the « mass » of the resonant state (*i.e.* total energy in c.m.s. $M \sim 159$ MeV $+ m + \mu$) and k_r is the c.m. momentum of the pion at the resonant energy.

The 3.3 contribution to σ_A, σ_B can then be written

$$\begin{cases} \sigma_A^+ = -2\sigma_A^- = (\varepsilon_{A_1} + \varepsilon_{A_2} t)\pi \, \delta(s - M^2) = G_A \pi \, \delta(s - M^2) , \\ \sigma_B^+ = -2\sigma_B^- = (\varepsilon_{B_1} + \varepsilon_{B_2} t)\pi \, \delta(s - M^2) = G_B \pi \, \delta(s - M^2) , \end{cases} \tag{3.2}$$

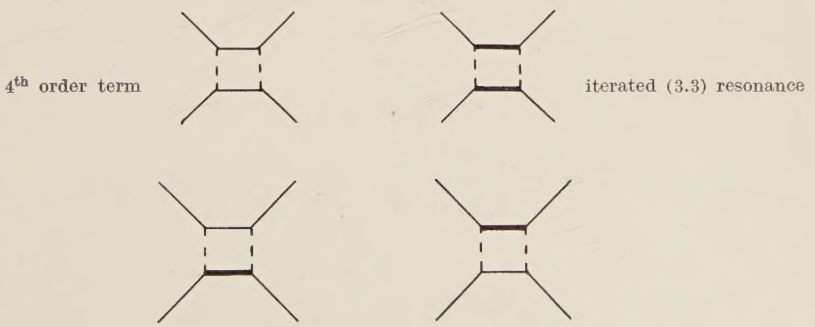
where

$$\begin{cases} \dfrac{\varepsilon_{A_1}}{4\pi} = \quad 7.4 , & \dfrac{\varepsilon_{A_2}}{4\pi} = 15 , \\ \dfrac{\varepsilon_{B_1}}{4\pi} = -17 , & \dfrac{\varepsilon_{B_2}}{4\pi} = 6.5 . \end{cases} \tag{3.3}$$

We shall now write each ϱ_j as

$$\varrho_j = \operatorname{Re} \varrho_j + i \operatorname{Im} \varrho_j \tag{3.4}$$

and consider separately the three types of terms in the ϱ_j. They are the 4-th order terms, the terms due to the iteration of the 3.3 resonance and the



mixed terms from iteration of (3.3) resonance with pole term.

Fig. 2.

« mixed » terms coming from the iteration of the 3.3 resonance with the pole term in $\bar{\sigma}_B$. (See Fig. 2). Note that since our entire method is only valid for

values of w for which inelastic processes are zero or very small (*i.e.* w not much larger than $(2m+\mu)^2$), we need keep only the 4-th order contribution to the imaginary parts of the ϱ_j, since the imaginary part contributed by the (σ_A, σ_B) is of the order of the square of the amplitude for the production process

$$\mathcal{N} + \mathcal{N} \to \mathcal{N} + \mathcal{N} + \pi \, .$$

In all cases we write the functions minus their s, p wave contributions in the $\mathcal{N}\overline{\mathcal{N}}$ channel.

3·1. *The 4-th order terms.*

a) The real part. Remembering that

$$\text{(3.5)} \qquad \begin{cases} t' = 2k^2(y-1)\,, \\ \bar{t}' = 4m^2 - t' - w\,, \\ k = \sqrt{\dfrac{w}{4} - m^2}\,, \\ q = \sqrt{\dfrac{t'}{4} - \mu^2}\,, \end{cases}$$

we define three functions $I^0(s', s'')$, $I'(s)$, $I''(s', s'')$ which arise naturally in the evaluation of the principal value integrals of (2.3) (*). These are

$$\text{(3.6)} \qquad I'(s) = \frac{2\pi}{q\varkappa} \operatorname{arctg} \mathcal{G}(s)$$

with

$$\text{(3.7)} \qquad \begin{cases} \varkappa = \sqrt{m^2 - \dfrac{t'}{4}} \\ \text{and} \\ \mathcal{G}(s) = \dfrac{2q\varkappa}{q^2 - \varkappa^2 + s}\,, \end{cases}$$

$$\text{(3.8)} \qquad I''(s', s'') = \frac{1}{t' - 4m^2} \Bigg\{ 4\pi(2w + t' - 4m^2) + \Bigg[(t' - 4m^2)(s' - s'') - 2w \left(\frac{t'}{2} - \mu^2 - m^2 + s'' \right) \Bigg] I'(s'') \Bigg\} \, .$$

(*) The explicit form of the $H_j(xt)$ of (2.2) was given in A.L.V. I, equations (3.36) to (3.39).

Introducing

$$
(3.9)\qquad \left\{\begin{aligned}
\xi(s's'') &= (s'-s'')\left(s''+\frac{t'}{2}-\mu^2-m^2\right)-\frac{w}{2}(t'-4\mu^2),\\
\eta(s's'') &= (s'-s'')^2+w(t'-4\mu^2),\\
\zeta(s'') &= \left(s''+\frac{t'}{2}-\mu^2-m^2\right)^2-\frac{1}{4}(t'-4\mu^2)(t'-4m^2),
\end{aligned}\right.
$$

we define

$$
(3.10)\qquad \left\{\begin{aligned}
I^0(s's'') &= \frac{2\pi}{\sqrt{\xi^2-\zeta\eta}}\log\left[\frac{\xi+\zeta+\sqrt{\xi^2-\zeta\eta}}{\xi+\zeta-\sqrt{\xi^2-\zeta\eta}}\right] && \text{for } \xi^2-\zeta\eta>0,\\
&= \frac{4\pi\eta}{\xi(\xi+\eta)}=\frac{4\pi}{\xi+\zeta} && \text{for } \xi^2-\zeta\eta=0,\\
&= \frac{4\pi}{\sqrt{\zeta\eta-\xi^2}}\operatorname{arctg}\left[\frac{\xi+\zeta}{\sqrt{\zeta\eta-\xi^2}}\right]^{-1} && \text{for } \xi^2-\zeta\eta<0.
\end{aligned}\right.
$$

It is also convenient to introduce the following functions:

$$
(3.11)\qquad \left\{\begin{aligned}
R(s's'') &= \frac{1}{2t'}\left\{I'(s')+I'(s'')-(t'+s'+s''-2\mu^2-2m^2)I^0(s's'')\right\},\\
S(s's'') &= \frac{1}{2w}\left\{I'(s')-I'(s'')+(s'-s'')\,I^0(s's'')\right\},\\
M(s's'') &= 2\pi+\tfrac{1}{4}\left[I''(s's'')+I''(s''s')\right]-\tfrac{1}{2}(s'+s''+t'-2\mu^2-2m^2)\cdot\\
&\qquad\cdot\left[I'(s')+I'(s'')\right]+\tfrac{1}{4}(s'+s''+t'-2\mu^2-2m^2)^2\,I^0(s's''),\\
X(s's'') &= -2\pi+\tfrac{1}{2}(s'-s'')\left[I'(s')-I'(s'')\right]+\\
&\qquad+\tfrac{1}{4}\left[I''(s's'')+I''(s''s')\right]+\tfrac{1}{4}(s'-s'')^2I^0(s's''),
\end{aligned}\right.
$$

$$
(3.12)\qquad \left\{\begin{aligned}
\gamma_1 &= \frac{3mg^2}{\varkappa^3 q}\left[\frac{3h}{2}-\frac{1+3h^2}{2}\operatorname{arctg}\frac{1}{h}\right] \quad (*),\\
\gamma_2 &= \frac{3}{2}\frac{1}{\varkappa^2 q}\left\{\frac{G_A}{q}\left[H\operatorname{arctg}\frac{1}{H}-1\right]-\frac{m}{\varkappa}G_B\left[\frac{3H}{2}-\frac{1+3H^2}{2}\operatorname{arctg}\frac{1}{H}\right]\right\},\\
\Gamma_1 &= \frac{g^2}{\varkappa q}\left[\frac{3h}{2}-\frac{(3+3h^2)}{2}\operatorname{arctg}\frac{1}{h}\right],\\
\Gamma_2 &= -\frac{1}{2\varkappa q}G_B\left[\frac{3H}{2}-\frac{3+3H^2}{2}\operatorname{arctg}\frac{1}{H}\right],
\end{aligned}\right.
$$

(*) Note that the functions α_0^+, α_1^-, β_1^- occurring in I in equations (3.28), (3.29) and (3.43) are very simply expressed in terms of these functions as

$$
\alpha_0^+=\frac{G_A}{\varkappa q}\operatorname{arctg}\frac{1}{H},\qquad \alpha_1^-=\gamma_1+\gamma_2,\qquad \beta_1^-=\Gamma_1+\Gamma_2.
$$

with

$$h=\frac{t'-2\mu^2}{4q\varkappa}\,;\qquad H=\frac{M^2-m^2+t'/2-\mu^2}{2q\varkappa}\,.$$

In general, when referring to some function of s' and s'', say $I^0(s's)$ we shall write its value at $s'=m^2$, $s''=M^2$ say as $I^0(m, M)$, *i.e.* without the square index.

Then, using for simplicity

$$N=\frac{1}{64\pi^2}\sqrt{t'-\frac{4\mu^2}{t'}}\,,$$

the functions $\varrho_j^{\pm}(t'w)$ become

$$\left\{\begin{aligned}
&\operatorname{Re}\varrho_1^+(w,t')=\frac{m^2Ng^4}{w(4m^2-t'-w)}\left\{\frac{3w+t'-4m^2}{4m^2-t'-w}\,M(mm)+\right.\\
&\qquad\left.+\frac{3w+2t'-8m^2}{w}\,X(mm)+\frac{1}{4}(t'-4\mu^2)(2w+t'-4m^2)\,I^0(mm)\right\},\\
&\operatorname{Re}\varrho_1^-(w,t')=\frac{\pi N}{24}(t'-4\mu^2)\,w\gamma_1^2-\operatorname{Re}\varrho_1^+(w,t')\,,\\
&\operatorname{Re}\varrho_2^+(w,t')=\frac{mNg^4}{w(4m^2-t'-w)}\left\{\frac{4m^2-t'+w}{4m^2-t'-w}\,M(mm)+\right.\\
&\qquad\left.+\frac{8m^2-2t'-w}{w}\,X(mm)+\frac{1}{4}(t'-4\mu^2)(4m^2-t')\,I^0(mm)\right\},\\
&\operatorname{Re}\varrho_2^-(w,t')=\frac{\pi N}{12}(t'-4\mu^2)\gamma_1\Gamma_1-\operatorname{Re}\varrho_2^+(w,t')\,,\\
&\operatorname{Re}\varrho_3^{\pm}(w,t')=\pm\frac{1}{m^2}\operatorname{Re}\varrho_1^+(wt')\,,\\
&\operatorname{Re}\varrho_4^+(w,t')=Ng^4\left\{\frac{t'-4\mu^2}{4}\,I^0(mm)+\frac{1}{4m^2-t'-w}\,M(mm)+\frac{1}{w}X(mm)\right\},\\
&\operatorname{Re}\varrho_4^-(w,t')=\frac{N\pi}{12}(t'-4\mu^2)\,\Gamma_1^2-\operatorname{Re}\varrho_4^+(wt')\,,\\
&\operatorname{Re}\varrho_5^{\pm}=0\,.
\end{aligned}\right.\tag{3.13}$$

b) The imaginary parts (*). Let us write the equation (2.8) for the boundary curve of the s', s'' integration as

$$\mathscr{F}(s's'')=\frac{x}{2}(t-4\mu^2)\,.\tag{3.14}$$

(*) Recall that these 4-th order terms are the whole contribution to the imaginary parts of the amplitude in our region of interest.

Then if

$$\frac{w}{2}(t-4\mu^2) < \mathscr{F}(mm), \tag{3.15}$$

all the imaginary parts of $\varrho_j^{\pm}$, and hence of the $\overline{p}_j^{\pm}$ of equation (2.13) are zero. [Note that since the imaginary part arises only in the function $\varrho_j^{\pm}(w, t')$ and not in the crossed term $\varrho_j^{\pm}(\bar{t}, t')$ we shall rather give here directly the formulae for the imaginary parts of the $\overline{p}_j^{\pm}(w, y)$.]

Otherwise the functions are:

$$\begin{cases} \operatorname{Im}\overline{p}_1^{\pm}(wy) = \pm \dfrac{m^2 g^4 L}{(w-4m^2+t')^2} \cdot \\ \qquad\qquad \cdot \left\{ \dfrac{4m^2-2w-t'}{w^2} K^{-1}(mm) + z^2(mm)\, K(mm) \right\}, \\ \operatorname{Im}\overline{p}_2^{\pm}(w, y) = \pm \dfrac{m g^4 L}{(w-4m^2+t')^2} \left\{ \dfrac{t'-4m^2}{w^2} K^{-1}(mm) + z^2(mm)\, K(mm) \right\}, \\ \operatorname{Im}\overline{p}_3^{\pm}(w, y) = \dfrac{1}{m^2} \operatorname{Im}\overline{p}_1^{\pm}(w, y), \\ \operatorname{Im}\overline{p}_4^{\pm}(w, y) = \pm \dfrac{g^4 L}{w(w-4m^2+t')} K^{-1}(mm), \\ \operatorname{Im}\overline{p}_5^{\pm} = 0, \end{cases} \tag{3.16}$$

where

$$z(mm) = t' - 2\mu^2$$

and

$$L = \frac{1}{32\pi\sqrt{t'}}$$

and one must of course write t' as a function of y (see equation (3.5)). Here $K(s's''tw)$ of I, (3.40), is taken at the value $x = w$, $s' = s'' = m^2$. Note that in terms of the functions defined earlier we have simply

$$K(s's'') = \frac{q}{\sqrt{\xi^2 - \zeta\eta}}.$$

These formulae, when used in eqs. (2.13), (2.12) and (2.11) allow us to evaluate the 4-th order contribution to the $T_{ij}(l)$.

Note that if one wishes to have the result of the full 4-th order perturbation terms (*i.e.* including its s, p wave projection in the $\mathcal{N}\overline{\mathcal{N}}$ channel), it is only necessary to disregard (*i.e.* set to zero) all terms in the above which contain explicitly γ_1, or Γ_1, since these are in fact just the subtracted s, p wave parts.

3·2. *The iterated* (3.3) *resonance.* – Considering just the δ functions at $s' = M^2$, $s'' = M^2$ in $\sigma_{A,B}$, the $\varrho_j^{\pm}$ functions become

$$
\left\{
\begin{aligned}
&\operatorname{Re}\varrho_1^+(t'w) = NG_A^2\left[I^0(MM) - \frac{\pi}{\varkappa^2 q^2}\left(\operatorname{arctg}\frac{1}{H}\right)^2\right] + \\
&\qquad + 2mNG_BG_A[R(MM) - S(MM)] + \operatorname{Re}\varrho_1^+(MM)\cdot\frac{G_B^2}{g^4}\,, \\
&\operatorname{Re}\varrho_1^-(t'w) = \frac{N\pi}{24}(t'-4\mu^2)w\gamma_2^2 - 2mN\frac{G_AG_B}{4}[R(MM) - S(MM)] - \\
&\qquad - \operatorname{Re}\varrho_1^+(MM)\frac{G_B^2}{4g^4} - \frac{NG_A^2}{4}I^0(MM)\,, \\
&\operatorname{Re}\varrho_2^+(t'w) = NG_AG_B[R(MM) + S(MM)] + \frac{G_B^2}{g^4}\operatorname{Re}\varrho_2^+(MM)\,, \\
&\operatorname{Re}\varrho_2^-(t'w) = \frac{N\pi}{12}(t'-4\mu^2)\gamma_2\Gamma_2 - \tfrac{1}{4}\operatorname{Re}\varrho_2^+(t'w)\,, \\
&\operatorname{Re}\varrho_3^+(t'w) = \binom{1}{\frac{1}{4}}\frac{G_B^2}{g^4}\operatorname{Re}\varrho_3^{\pm}(MM)\,, \\
&\operatorname{Re}\varrho_4^+(t'w) = \frac{G_B^2}{g^4}\operatorname{Re}\varrho_4^+(MM)\,, \\
&\operatorname{Re}\varrho_4^-(t'w) = \frac{N\pi}{12}(t'-4\mu^2)\Gamma_2^2 - \tfrac{1}{4}\operatorname{Re}\varrho_4^+(t'w)\,, \\
&\operatorname{Re}\varrho_5^{\pm} = 0\,,
\end{aligned}
\right. \tag{3.17}
$$

where by $\varrho_j^{\pm}(MM)$ we mean the same function as used in the 4-th order result but evaluated at $s' = M^2$, $s'' = M^2$.

3·3. *The « mixed » terms.* – Finally we write down the contributions arising from the mixed terms of the 3,3 resonance and Born terms

$$
\left\{
\begin{aligned}
&\operatorname{Re}\varrho_1^+(t'w) = 2mNg^2G_A[R(Mm) - S(Mm)] + \\
&\qquad + \frac{m^2G_B}{g^2}\{\operatorname{Re}\varrho_1^+(mM) + \operatorname{Re}\varrho_1^+(Mm)\}, \\
&\operatorname{Re}\varrho_1^-(t'w) = \frac{\pi N}{12}(t'-4\mu^2)w\gamma_1\gamma_2 + \tfrac{1}{2}\operatorname{Re}\varrho_1^+(t'w)\,, \\
&\operatorname{Re}\varrho_2^+(t'w) = Ng^2G_A[R(Mm) + S(Mm)] + \\
&\qquad + \frac{mG_B}{g^2}\{\operatorname{Re}\varrho_2^+(mM) + \operatorname{Re}\varrho_2^+(Mm)\}, \\
&\operatorname{Re}\varrho_2^-(t'w) = \frac{\pi N}{12}(t'-4\mu^2)(\gamma_1\Gamma_2 + \gamma_2\Gamma_1) + \tfrac{1}{2}\operatorname{Re}\varrho_2^+(t'w)\,,
\end{aligned}
\right. \tag{3.18}
$$

$$
(3.18)\qquad
\begin{cases}
\operatorname{Re}\varrho_3^{\pm}(t'w) = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \dfrac{G_B}{g^2}\left\{\operatorname{Re}\varrho_3^{\pm}(mM) + \operatorname{Re}\varrho_3^{\pm}(Mm)\right\}, \\
\operatorname{Re}\varrho_4^{+}(t'w) = \dfrac{G_B}{g^2}\left\{\operatorname{Re}\varrho_4^{+}(mM) + \operatorname{Re}\varrho_4^{+}(Mm)\right\}, \\
\operatorname{Re}\varrho_4^{-}(t'w) = \dfrac{\pi N}{6}(t' - 4\mu^2)\Gamma_1\Gamma_2 + \tfrac{1}{2}\operatorname{Re}\varrho_4^{+}(t'w), \\
\operatorname{Re}\varrho_5^{\pm} = 0,
\end{cases}
$$

where by $\varrho_j^{\pm}(mM)$ say we mean the 4-th order function used with $s' = m^2$, $s'' = M^2$.

3·4. We now have a complete specification of all the $\varrho_j^{\pm}$ functions arising from the iteration of the Born plus (3.3) resonance terms of pion-nucleon scattering, with the s and p wave projections of these terms in the $N\bar{N}$ channel explicitly subtracted out. Each of the $\varrho_j^{\pm}$ functions to be used in (2.13) is of course a sum of the $\varrho_j^{\pm}$ arising from the three types of contributions discussed above.

In the next section, then, we shall consider specific models of the s or p wave π-π interaction, which will then contribute a further term to the $\varrho_j(wt)$ (see equation (2.3)).

Finally, for easy reference, we give the matrices $U_{\beta\alpha}$ and V_{ij}^{α}:

$U_{\alpha\beta} =$

	$\beta=1$	$\beta=2$	$\beta=3$	$\beta=4$	$\beta=5$
$\alpha=1$	1	$(1/m)[2m^2+y(4m^2-w)]$	$m^2-w+y(4m^2-w)$	-4	$+1$
2	1	$-2m$	$w-3m^2$	2	-1
3	1	$2m$	m^2-w	-2	-1
4	1	$(1/m)(2m^2-w)$	m^2	0	$+1$
5	1	$-(1/m)[6m^2-y(4m^2-w)]$	$m^2+w-y(4m^2-w)$	4	$+1$

$\times\left(\frac{1}{4}\right)$

$V_{ij}^{\alpha} =$

	$\alpha=1$	$\alpha=2$	$\alpha=3$	$\alpha=4$	$\alpha=5$
$ij=11$	0	$\lambda[2\lambda-(\lambda-1)(1-y^2)]$	$2y(1-\lambda^2)$	$2+(\lambda-1)(1-y^2)$	0
$1-1$	0	$\lambda(\lambda-1)$	0	$1-\lambda$	0
(10)—(01)	$\sqrt{2}(\lambda^2-1)$	$-\sqrt{2}\,y(\lambda-1)^2$	$\sqrt{2}(\lambda^2-1)$	$-\sqrt{2}\,y(\lambda-1)^2$	0
00	$2y(1-\lambda^2)$	$2[\lambda-y^2(\lambda-1)]$	0	$2\lambda[1+y^2(\lambda-1)]$	0
ss	0	0	2	$2y(1-\lambda^2)$	$2\lambda^2$

$\times\left(\frac{m}{4\pi\lambda}\right)$

with $\lambda = \dfrac{E}{m} = \dfrac{\sqrt{w}}{2m}$.

4. -- Contribution of 2π s and p waves of the $\mathcal{N}\overline{\mathcal{N}} \to 2\pi$ process.

4·1. *Relation with pion-pion scattering.* – Let us discuss now the information we can obtain on η_1^-, λ_0^+ and λ_1^-, that are related to the s and p contribution to the amplitudes for the process $\mathcal{N}\overline{\mathcal{N}} \to 2\pi$ through (3.32) of I.

We shall refer here to the analysis (and notation) of FRAZER and FULCO [4] and B.C.L. [5] papers. With the usual definition of amplitudes with definite total angular momentum J and with definite helicity, we obtain

$$\text{(4.1)} \qquad \begin{cases} A_{sp}^{(\pm)} = -\dfrac{4\pi}{p^2} f_+^{0(\pm)}(t) + 12\pi \dfrac{q}{p} \cos\theta \left(\dfrac{m}{\sqrt{2}} f_-^{1(\pm)} - f_+^{1(\pm)} \right) \\ B_{sp}^{(\pm)} = \dfrac{12\pi}{\sqrt{2}} f_-^{(1\pm)} . \end{cases}$$

We see then that λ_0^+, λ_1^-, η_1^- are given by

$$\text{(4.2)} \qquad \begin{cases} \lambda_0^+(t) = -\dfrac{4\pi}{p^2} f_+^{0-}(t) , \\ \lambda_1^-(t) = \dfrac{12\pi}{p^2} \left(\dfrac{m}{\sqrt{2}} f_-^{1-}(t) - f_+^{1-}(t) \right) , \\ \eta_1^-(t) = \dfrac{12\pi}{\sqrt{2}} f_-^{1-}(t) . \end{cases}$$

The analysis of the $\mathcal{N}\overline{\mathcal{N}} \to 2\pi$ process must be carried out in order to get information on the functions $f^J(t)$. Because they represent the whole s and p amplitude for the process (this is the reason for the subdivision (3.31) of I), unitarity can be imposed on them by requiring their phase to be that of the π-π scattering with the same quantum numbers. Using the analytic properties of the function $f^J(t)$ obtained from a Mandelstam representation analysis and the unitarity requirements (in the region of t we are interested in), the following solution can be obtained [4]

$$\text{(4.3)} \qquad f_\pm^J(t) = f_{\pi\pi}^J(t) \int_{-\infty}^{a} \frac{\operatorname{Im} f_\pm^J(t')\, \mathrm{d}t'}{(t' - t - i\varepsilon) f_{\pi\pi}^J(t')} ,$$

where

$$\text{(4.4)} \qquad f_{\pi\pi}^J = \frac{\exp\left[i\delta_{\pi\pi}^J\right] \sin \delta_{\pi\pi}^J}{q^{2J+1}} ,$$

with $\delta_{\pi\pi}^J$ the phase shift for π-π scattering with angular momentum J. The upper limit of integration in (4.3) is given by

$$\text{(4.5)} \qquad a = 4\mu^2 \left(1 - \frac{\mu^2}{4m^2} \right) .$$

We realize now that we are interested in values of $\lambda(t)$ and $\eta(t)$ (and therefore of $f(t)$) for values of t greater than $4\mu^2$. This means that the integral in (4.3) having a cut only for $t > a$ will give rise to a function probably weakly dependent on t. So even without requiring further analysis of $\mathcal{N}\bar{\mathcal{N}} \to 2\pi$, so as to determine $f^J_{\pm}$ from (4.3), we expect that the main feature for the t dependence of $f^J_{\pm}$ will be given by $f^J_{\pi\pi}(t)$. Unfortunately, even this feature is not well known at present; there are indications however coming from several analyses, and hope that more knowledge will soon be forthcoming. However, the characteristic pointed out before, *i.e.* that in our region of interest we shall have approximately

$$f^J_{\pm}(t) \propto f^J_{\pi\pi}(t)\,, \tag{4.6}$$

will allow us to try various models in order to evaluate λ_0, λ_1 and η_1 from (4.2) in terms of some constants that will appear also in analogous models applied to other physical processes. Thus we shall try to perform the analysis for a sharp p wave pion-pion resonance ($J = T = 1$): there is already some convincing indication for such a behaviour, coming from electromagnetic nuclear structure ([4]), d-wave pion nucleon scattering ([5]) and analysis of higher resonances. Better analysis in these directions will allow the clearing up of the existence of the π-π resonance, and determine its parameters (position, width, etc.). In Section 3·2 we shall give the contribution of such resonance to $\mathcal{N}$-$\mathcal{N}$ scattering as functions of these parameters.

4·2. $T = J = 1$ π-π *resonance model.* – We shall evaluate here the contributions to $\mathcal{N}$-$\mathcal{N}$ scattering coming from the integrals over $|\lambda|^2$ and $|\eta|^2$ in (3.7), (3.43) of I in a model of sharp pion pion resonance in p wave. We shall disregard, therefore for the moment, λ_0^+.

We shall adopt for $f^1_{\pi\pi}(t)$ a Breit-Wigner form satisfying the unitarity condition

$$\operatorname{Im} \frac{1}{f^1_{\pi\pi}} = -q^3\,, \tag{4.7}$$

that is

$$f^1_{\pi\pi}(t) = \frac{\gamma}{(t_r - t) - i\gamma q^3}\,, \tag{4.8}$$

where t_r is the square of the total energy at the resonance.

The consequences of (4.8) with the simplifying assumption of a very small γ (narrow resonance) on the isovector electromagnetic structure of nucleons and π-$\mathcal{N}$ scattering have been analyzed in detail by B.C.L.; they showed how results can be explicitly given in terms of some constants C_1, C_2 (form. (4.12),

(4.13), (5.13) of B.C.L.) and γ. The comparison with experiments (*) has yielded an excellent fit of the theory; the best value of the parameters being given by:

$$(4.9)\qquad \begin{cases} t_r = 22.4\mu^2\,, \\ C_1 = -1.0\,, \\ C_2 = -0.272\mu^{-1}\,, \\ \gamma = 0.376\mu^{-1}\,. \end{cases}$$

From (4.8) and (4.6) we can write approximately

$$(4.10)\qquad f^1_{\pm}(t) = \frac{N_{\pm}}{t_r - t - i\gamma q^3}\,,$$

so that

$$(4.11)\qquad \eta_1^-(t) = \frac{M_1}{t_r - t - i\gamma q^3}\,, \qquad \lambda_1^-(t) = \frac{M_2}{t_r - t - i\gamma q^3}\,,$$

with

$$(4.12)\qquad \begin{cases} M_1 = \dfrac{12\pi}{\sqrt{2}}\, N_-\,, \\ M_2 = \dfrac{12\pi}{p^2}\left(\dfrac{m}{\sqrt{2}}\, N_- - N_+\right). \end{cases}$$

From (4.5) and in the case of a very sharp resonance

$$(4.13)\qquad \begin{cases} |\eta_1^-|^2 = \dfrac{M_1^2}{(t_r - t)^2 + \gamma^2 q^6} \sim \dfrac{M_1^2}{\Gamma}\pi\delta(t_r - t)\,, \\ |\lambda_1^-|^2 = \dfrac{M_2}{(t_r - t)^2 + \gamma^2 q^6} \sim \dfrac{M_2^2}{\Gamma}\pi\delta(t_r - t)\,, \\ \mathrm{Re}\,(\lambda_1^{-*}\eta_1^-) = \dfrac{M_1 M_2}{(t_r - t)^2 + \gamma^2 q^6} \sim \dfrac{M_1 M_2}{\Gamma}\,\pi\,\delta(t_r - t)\,, \end{cases}$$

where the coefficient of the δ function is calculated in such a way that the two expressions for the resonance subtend the same area. Γ is then given by

$$(4.14)\qquad \Gamma = \gamma q_r^3\,,$$

where

$$(4.15)\qquad q_r = \sqrt{\frac{t_r}{4} - \mu^2}\,,$$

(*) J. Bowcock, W. N. Cottingham and D. Lurié: *Nuovo Cimento*, in press and *Phys. Rev. Lett.*, in press.

M_1 and M_2 are related to the numbers C_1 and C_2 of B.C.L. (see formulae following (4.13) in B.C.L.) by

$$\text{(4.16)} \qquad \begin{cases} M_1 = -12\pi(1 + 2mC_2), \\ M_2 = 24\pi C_2 . \end{cases}$$

Then, from (4.13), and (3.43) and (3.7) of I the contributions of the π-π p-wave resonance to the $\bar{p}_i^{\pm}$ are given by

$$\text{(4.17)} \quad \begin{cases} {}^{(\pi-\pi)}\bar{p}_1^{-}(wy) = \dfrac{12}{k^2}\pi^3(\bar{t}' - w)t\sqrt{t_r\left(\dfrac{t_r}{4} - \mu^2\right)^3}\dfrac{C_2^2}{\Gamma}\delta\left(y - 1 - \dfrac{t_r}{2k^2}\right)\dfrac{1}{2(2\pi)^2 t}, \\ {}^{(\pi-\pi)}\bar{p}_2^{-}(wy) = \dfrac{24}{k^2}\pi^3\sqrt{t_r\left(\dfrac{t_r}{4} - \mu^2\right)^3}\dfrac{C_2(C_1 + 2mC_2)}{\Gamma}\delta\left(y - 1 - \dfrac{t_r}{2k^2}\right)\dfrac{1}{2(2\pi)^2 t}, \\ {}^{(\pi-\pi)}\bar{p}_4^{-}(wy) = \dfrac{12}{k^2}\pi^3\sqrt{t_r\left(\dfrac{t_r}{4} - \mu^2\right)^3}\dfrac{(C_1 + 2mC_2)^2}{\Gamma}\delta\left(y - 1 - \dfrac{t_r}{2k^2}\right)\dfrac{1}{2(2\pi)^2 t}, \end{cases}$$

the contribution to the remaining $\bar{p}_i^{\pm}$ being zero.

From (4.17) the contribution to the $T_{ij}(l)$ are directly calculable by the same method used in Section **3**.

4·3. – In order to have a simple insight into the connection of the π-π p-wave resonance with other phenomena, we note that a δ-form resonance presents itself in a dispersive approach as a « stable » intermediate bound state. Let us call $\Pi\Pi$ this « bound state » of two pions in a p-wave state that will present itself as a vector-isovector particle with mass t_r. If such a « particle » would be present it would give a contribution to the dispersion relation (3.7) of I of the form $R_j/(t_r - t)$, where the R_j would be the projection over the appropriate invariant given by computation of graph (a) in which the two nucleons exchange a $\Pi\Pi$. It is easy to see that the results (4.17). have just this form; in fact they happen to be exactly the contribution of graph (a) calculated with an interaction hamiltonian between $\Pi\Pi$ and nucleons proportional to

Graph (a).

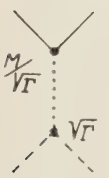
$$\text{(4.18)} \qquad i\frac{M_2}{\sqrt{\Gamma}}\bar{\psi}(p')\gamma_\mu\psi(p)\,\Phi^\mu_{\Pi\Pi} + \frac{M_1}{\sqrt{\Gamma}}\bar{\psi}(p')\,\psi(p)\,\Phi^\mu_{\Pi\Pi}\frac{(p + p')_\mu}{2}$$

(there are two coupling constants because of the two different couplings that a vector meson can have with a spinor field).

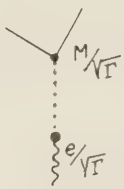
It is easy to see that in phenomena such as electromagnetic form factor of nucleons and pions, pion-pion scattering, π-$\mathcal{N}$ scattering, etc., the contribution of a π-π sharp resonance give equivalent results to those obtained by a lowest order perturbative result with an intermediate Π that has coupling constants with the different fields proportional to

$$\begin{cases} \sqrt{\Gamma} & \text{for } (\pi\,\Pi\,\pi)\,, \\ e/\sqrt{\Gamma} & \text{for } (\gamma\,\Pi)\,, \\ \dfrac{M_1}{\sqrt{\Gamma}},\ \dfrac{M_2}{\sqrt{\Gamma}} & \text{for } (\mathcal{N}\,\Pi\,\mathcal{N})\,. \end{cases} \tag{4.19}$$

From (4.19) it is easy to understand that since the Π contribution to π-$\mathcal{N}$ scattering is proportional to M_1, M_2 (graph (*b*)) and to the nucleon electromagnetic structure proportional to eM_1/Γ, eM_2/Γ, (graph (*c*)) an analysis of such contribution will allow an evaluation of M_1, M_2, t_r and Γ and then a determination of the corresponding contribution to $\mathcal{N}$-$\mathcal{N}$ scattering given in (4.17). So far we said nothing about λ_0^+, it is related through (4.2) to s wave π-π scattering for which there is no much evidence up to now. Some knowledge on such s-wave π-π interaction may, however, be forthcoming in the near future by π-$\mathcal{N}$ scattering analysis as well as τ-decay ([8]) and, possibly, the double π production by pions in the Coulomb field of the nucleus as recently proposed by FERRETTI ([9]).



Graph (*b*).



Graph (*c*).

5. – Conclusions.

In the previous sections we have given an explicit specification of the single and triplet partial wave amplitudes valid for the high waves. The results have been presented in such a way as to show explicitly the contribution arising from the effect of the Born term in π-$\mathcal{N}$ scattering [*i.e.* 4-th order term in $\mathcal{N}$-$\mathcal{N}$ scattering: see Section 3·1], the (3.3) π-$\mathcal{N}$ resonance (Sections 3·2 and 3·3) and the effect of the π-π interaction (Section 4). This final contribution has been tentatively calculated in the case of π-π p-wave resonance, in terms of the parameters characterizing such a resonance.

([8]) N. N. KHURI and S. B. TREIMAN: *Phys. Rev.*, **119**, 1115 (1960); R. F. SAWYER and K. L. WALI: *Phys. Rev.*, **119**, 1429 (1960).

([9]) B. FERRETTI: CERN seminar (1959).

The final integration (see eq. (3.1)) is at present being carried out numerically on the CERN Mercury computer and it is hoped to be able to give soon a comparison between the results of the above theory and experimental data.

We want to stress finally that the entire method relies on the fact that the higher singularities in t (3π or more massive intermediate states) do not give rise to anomalously big contributions. The latter situation could be caused, for instance, by the existence of a strong 3π resonance as recently proposed ([10]) and the extra contribution of this could then be added to our results. A way of adding it would be to perform a Born calculation with such a « bound state » in a manner similar to that explained in Section 4·3 for the $J = T = 1$ π-π resonance ([11]).

* * *

We are grateful to Professor S. Fubini for his continued interest and encouragement.

([10]) G. E. Chew: *Phys. Rev. Lett.*, **4**, 142 (1960).

([11]) Such a calculation for an intermediate « bound 3π state » with $T = 0$, $J = 1$ has been recently performed for the function p_2 (that gives rise to $S \cdot L$ potential) by J. J. Sakurai: *Phys. Rev.*, **119**, 1784 (1960).

RIASSUNTO

La teoria dell'interazione nucleone-nucleone sviluppata in un precedente lavoro ([1]), viene applicata in questo alle onde di alto momento orbitale. Queste onde sono quelle per le quali il contributo degli stati intermedi con tre mesoni π (o stati con massa ancora più elevata) è trascurabile. In questo caso, la rappresentazione data in I si riduce a delle espressioni analitiche per le ampiezze delle diverse onde parziali. Queste formule contengono esplicitamente le caratteristiche della diffusione pione-nucleone (costante d'accoppiamento, correzioni dovute alla risonanza (3, 3), ecc.) così come l'effetto dell'interazione pione-pione. Quest'ultimo effetto è calcolato esplicitamente per il caso di una risonanza π-π nello stato $T = J = 1$, in funzione di alcuni parametri che caratterizzano questa risonanza. Benchè l'effettiva distinzione tra onde « basse » ed « elevate » debba essere determinata da un confronto con l'esperienza, le formule ottenute sono probabilmente valide per $l \geqslant 2$.

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The Two Meson Approximation for the Real Nucleon.

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Summary. — The method of momenta proposed by F. Halpern [1,2] is applied to the description of the real nucleon state in the two meson approximation. The static model of Chew and Low for an extended source has been used. It seems possible to get a weakly coupled model which gives a good value of f^2 and which fits rather well the sum rules for the pion-nucleon scattering amplitudes.

1. – Introduction and notation.

We start with the Hamiltonian

$$H = H_{\mathrm{I}} + H_{\mathrm{II}} . \tag{1}$$

Here H_{I} is the Hamiltonian of the pions

$$H_{\mathrm{I}} = \frac{1}{2} \sum_{\alpha=1}^{3} \int \mathrm{d}^3x \{\pi_\alpha^2 + (\nabla \Phi_\alpha)^2 + \Phi_\alpha^2\} , \tag{2}$$

H_{II} is the interaction Hamiltonian between the pions and a static and extended nucleon

$$H_{\mathrm{II}} = \sqrt{4\pi} f_0 \int \mathrm{d}^3x \, U(\boldsymbol{x})(\boldsymbol{\sigma}\nabla) \sum_{\alpha=1}^{3} (\tau_\alpha \Phi_\alpha) . \tag{3}$$

(1) F. R. Halpern: *Phys. Rev.*, **107**, 1145 (1957).

(2) F. R. Halpern, L. Sartori, K. Nishimura and R. Spitzer: *Ann. Phys.*, **7**, 154 (1959).

We put $M_\pi = \hbar = c = 1$. f_0 is the unrenormalized coupling constant. $U(\boldsymbol{x})$ is the nucleon source function, which is normalized to one:

$$\int \mathrm{d}^3x\, U(\boldsymbol{x}) = 1\,. \tag{4}$$

$$v(k) = \int \mathrm{d}^3x \exp\,[-i\boldsymbol{k}\boldsymbol{x}]\, U(\boldsymbol{x})\,, \qquad (k = |\boldsymbol{k}|), \tag{5}$$

is the Fourier transform of $U(x)$. On account of (4) we have

$$v(0) = 1\,,$$

σ_α, τ_β are the spin and isospin matrices of the nucleon. We introduce the charged fields

$$\varphi_\pm = \tfrac{1}{2}(\mp\varphi_1 + i\varphi_2) \qquad \text{and} \qquad \varphi_0 = \varphi_3$$

and expand them in terms of angular momentum eigenfunctions, normalized to a sphere of radius R:

$$\varphi_\pm(\boldsymbol{x}) = \sum_k \{a_{\pm,k}\psi_k(\boldsymbol{x}) - a^\dagger_{\mp,k}\psi^*_k(\boldsymbol{x})\}\,,$$

$$\varphi_0(\boldsymbol{x}) = \sum_k \{a_{0,k}\psi_k(\boldsymbol{x}) + a^\dagger_{0,k}\psi^*_k(\boldsymbol{x})\}\,,$$

$$\pi_\pm(\boldsymbol{x}) = i\sum_k \omega_k\{a_{\mp,k}\psi_k(\boldsymbol{x}) + a^\dagger_{\pm,k}\psi^*_k(\boldsymbol{x})\}\,,$$

$$\pi_0(\boldsymbol{x}) = -i\sum_k \omega_k\{a_{0,k}\psi_k(\boldsymbol{x}) - a^\dagger_{0,k}\psi^*_k(\boldsymbol{x})\}\,.$$

Here k stands for (n, l, m),

$$\psi_k(\boldsymbol{x}) = P_n(R\omega_n)^{-\frac{1}{2}} j_l(P_n r)\, Y_{lm}(\vartheta, \varphi)\,,$$

where $\omega_n = (1 + P_n^2)^{\frac{1}{2}}$ and $P_n = (\pi/2R)(2n + l)$,

$$[a_{ik}, a_{i'k'}]_- = 0\,; \quad [a_{ik}, a^\dagger_{i'k'}]_- = \delta_{ii'}\delta_{kk'} \quad (i, i', k, k' = -, 0, +)\,.$$

With these relations we get from (1), (2), and (3)

$$H = \sum_k a^\dagger_k a_k \omega_k + \sum_n V_n(A_n + A^\dagger_n)\,. \tag{6}$$

Here V_n means

$$V_n = f_0(3R\omega_n)^{-\frac{1}{2}} P_n^2 v(P_n) \tag{7}$$

and

$$A_n = \frac{1}{\sqrt{2}}\sigma_-\left(\frac{1}{\sqrt{2}}\tau_- a^n_{--} - \frac{1}{\sqrt{2}}\tau_+ a^n_{+-} + \tau_3 a^n_{0-}\right) -$$

$$-\frac{1}{\sqrt{2}}\sigma_+\left(\frac{1}{\sqrt{2}}\tau_- a_{-+} - \frac{1}{\sqrt{2}}\tau_+ a^n_{++} + \tau_3 a^n_{0+}\right) + \sigma_3\left(\frac{1}{\sqrt{2}}\tau_- a^n_{-0} - \frac{1}{\sqrt{2}}\tau_+ a^n_{+0} + \tau_3 a^n_{00}\right),$$

a^n_{ij} are the pion destruction operators for $l = 1$, isospin $I_3 = i$, and angular momentum $L_3 = j$. If we write

$$(8) \qquad \varphi_{p\uparrow} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad \varphi_{p\downarrow} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \qquad \varphi_{n\uparrow} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \qquad \varphi_{n\downarrow} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix},$$

for the charge and spin states of the bare nucleon, the A_n take the form

$$(9) \qquad A_n = \begin{pmatrix} a^n_{00} & -\sqrt{2}\,a^n_{0+} & -\sqrt{2}\,a^n_{+0} & 2\,a^n_{++} \\ \sqrt{2}\,a^n_{0-} & -a^n_{00} & -2\,a^n_{+-} & \sqrt{2}\,a^n_{+0} \\ \sqrt{2}\,a^n_{-0} & -2\,a^n_{-+} & -a^n_{00} & \sqrt{2}\,a^n_{0+} \\ 2\,a^n_{--} & -\sqrt{2}\,a^n_{-0} & -\sqrt{2}\,a^n_{0-} & a^n_{00} \end{pmatrix}.$$

2. – The variation method.

The state of the real nucleon we approximate by the state vector

$$10) \qquad \psi_n = \sum_{k=0}^{n} \alpha_k \varphi_k\,, \qquad \varphi_k = H^k \varphi\,,$$

ψ_n contains the bare nucleon state φ and states with 1, 2, ..., n pions. The numbers α_k are determined by a variation method such as to minimize the energy

$$(11) \quad E(\alpha_0 \ldots \alpha_n) = (\psi_n, H\psi_n)/(\psi_n, \psi_n) = \sum_{k=0}^{n}\sum_{k'=0}^{n} \alpha_k \alpha_{k'} H_{k+k'+1} \Big/ \sum_{l=0}^{n}\sum_{l'=0}^{n} \alpha_l \alpha_{l'} H_{l+l'}\,.$$

We introduced the momenta

$$H_k = (\varphi, H^k\varphi) = (\varphi_1, \varphi_k)\,.$$

The H_k are expressible by the integrals

$$I_m = \sum_n V_n^2 \omega_n^{m-2} \to \frac{f_0^2}{3\pi} \int_0^\infty \mathrm{d}k\, k^4 v^2(k) \omega_k^{m-3}, \tag{12}$$

for $R \to \infty$. So we have

$$H_0 = 1\,, \quad H_1 = 0\,, \quad H_2 = 9I_2\,, \quad H_3 = 9I_3\,, \quad \ldots\,.$$

The dependece of the higher H_k's on the I_m's is given in the paper of HALPERN *et al.* [2]. The variation of (11) leads to the system

$$\sum_{k'=0}^{n} \alpha_{k'}(H_{k+k'+1} - EH_{k+k}) = 0\,; \qquad k = 0, 1, 2, \ldots, n. \tag{13}$$

The equations (13) have a non-trivial solution, if the determinant vanishes, that means if

$$\begin{vmatrix} 1 & E & \ldots & E^{n+1} \\ H_0 & H_1 & \ldots & H_{n+1} \\ \cdot & \cdot & \cdot & \cdot \\ H_n & H_{n+1} & \ldots & H_{2n+1} \end{vmatrix} = 0 \tag{14}$$

holds. The lowest eigenvalues of the $n+1$ roots of (14) is the best approximation to the energy of the real nucleon in the n-pion approximation. The condition (14) is achieved too, if we start with the Schrödinger-equation

$$H\psi_n = E\psi_n \tag{15}$$

and project this equation into the $n+1$ linear independent states φ, φ_1, φ_2, ..., φ_n:

$$(\varphi_k, H\psi_n) = (\varphi_k, E\psi_n)$$

or

$$\sum_{k'=0}^{n} \alpha_{k'}(H_{k+k'+1} - E^{k+1}H_{k'}) = 0\,; \qquad k = 0, 1, 2, \ldots, n.$$

Of course it is impossible to satisfy (15) exactly in the n-pion approximation, as

$$H\psi_n = \sum_{k=0}^{n} \alpha_k \varphi_{k+1}$$

also contains a contribution $\alpha_n\varphi_{n+1}$ which is a state with $n+1$ mesons. α_n^2 is thus a measure for the error we take in the n-pion approximation. If $\lim_{n\to\infty}\alpha_n = 0$, the method should converge. HALPERN *et al.* could show for a density function

$$v(\boldsymbol{k}) = \begin{cases} 1 & \\ 0 & \end{cases} \qquad \text{for } |\boldsymbol{k}| \lesssim M,$$

that the physical quantities of the theory converge very soon.

To get a reasonable source distribution in x-space we choose

$$U(\boldsymbol{x}) = (2\pi\alpha)^{-\frac{3}{2}} \exp\left[\frac{-r^2}{2\alpha}\right], \qquad r = |\boldsymbol{x}|, \qquad v(k) = \exp\left[-\frac{\alpha}{2}k^2\right]. \tag{16}$$

Yukawa-form distributions are excluded by the fact that the n-pion approximation contains H_0, H_1, ..., H_{2n+1} and therefore the integrals I_m with $m \leqslant 2n+1$. Thus in the n-pion approximation all integrals I_m up to

$$\int_0^\infty \mathrm{d}k\, k^4 v^2(k)(1+k^2)^{n-1},$$

must converge; this means: the relation

$$v^2(k)k^{2(n+1)} \to 0, \qquad \text{for } k \to \infty,$$

must hold. This is not the case for a Yukawa-form distribution

$$v(k) = (1 + k^2/M^2)^{-1}.$$

We define the constants

$$r_1 = \frac{(\psi_n, \sigma_j\psi_n)}{(\Phi, \sigma_j\Phi)} = \frac{(\psi_n, \tau_j\psi_n)}{(\Phi, \tau_j\Phi)}, \tag{17}$$

$$r_2 = \frac{(\psi_n, \sigma_i\tau_j\psi_n)}{(\Phi, \sigma_i\tau_j\Phi)} = \frac{f}{f_0}, \tag{18}$$

where f is the renormalized coupling constant. (17) holds because the spin and the isospin of the nucleon enters the theory in a symmetric way. By use of (17) we get for the charge Q_π of the pion cloud:

$$Q_\pi = \left(\psi_n, \left(1 - \frac{1+\tau_3}{2}\right)\psi_n\right)\varepsilon = \frac{1}{2}(1 - r_1)\varepsilon. \tag{19}$$

Now we want to take into account the strong sum rules for the fixed source meson theory derived by CINI, FUBINI and THIRRING ([3-6]). For example Q_π is given by the integral ([5])

$$Q_\pi = \frac{2\varepsilon}{\pi}\int_0^\infty \mathrm{d}k\,\frac{k^4}{\omega^4}\,v^2(k)\,\frac{\mathrm{d}}{\mathrm{d}\omega_k}\left\{-\frac{f^2}{\omega_k}-\beta_{11}(\omega_k)-\beta_{13}(\omega_k)+2\beta_{33}(\omega_k)\right\},$$

where

$$\beta_{ik}(\omega_k)=\beta_{ki}(\omega_k)=\frac{1}{3\pi}\int_1^\infty \frac{\mathrm{d}\omega_p\,\sin^2\delta_{ik}(\omega_p)}{p^3 v^2(p)(\omega_p+\omega_k)}\,.$$

The $\delta_{I,J}$'s are the (real) phase shifts for pion nucleon scattering in the states with isospin I and angular momentum J. Neglecting β_{11}, β_{13}, and β_{31} we get

$$(20)\quad r_1 = 1-\frac{2}{\varepsilon}\,Q_\pi \approx 1-\frac{4}{\pi}\int_0^\infty \frac{\mathrm{d}k\,k^4}{\omega_k}\,v^2(k)\left\{\frac{f^2}{\omega_k^2}-\frac{2}{3\pi}\,(\omega(p_r)+\omega_k)^{-2}\int_1^\infty \frac{\mathrm{d}\omega_p\,\sin^2\delta_{33}(\omega_p)}{p^3 v^2(p)}\right\},$$

if we assume that the main contribution comes from the resonance at the pion energy

$$\omega(p_r)=(1+p_r^2)^{\frac{1}{2}}\,.$$

FUBINI and THIRRING ([6]) by discussion of their sum rules suggested for f_0 and r_1 the following values:

$$f_0^2 = 0.22\,, \qquad r_1 = 0.37 \pm 0.01\,.$$

If we use $r_1 = 0.37$ and $f^2 = 0.08$ the numerical integration of (20) yields for a density function (16) the value

$$(21)\qquad \alpha = 0.049\,.$$

3. – The two meson approximation.

With (21) and $f_0^2 = 0.22$ the integrals

$$I_m = (f_0^2/3\pi)\int_0^\infty \mathrm{d}k\,k^4\,\exp\left[-\alpha k^2\right](1+k^2)^{(m-3)/2}\,,$$

(3) M. CINI and S. FUBINI: *Phys. Rev.*, **102**, 1687 (1956).
(4) M. CINI and S. FUBINI: *Nuovo Cimento*, **3**, 764 (1956).
(5) S. FUBINI: *Nuovo Cimento*, **3**, 1425 (1956).
(6) S. FUBINI and W. E. THIRRING: *Phys. Rev.*, **105**, 1382 (1957).

become numerically

$$I_1 = 0.8886\,, \qquad I_4 = 200.88\,,$$
$$I_2 = 4.7383\,, \qquad I_5 = 1518.4\,,$$
$$I_3 = 29.189\,.$$

This gives

$$H_0 = 1\,, \qquad H_3 = 262.70\,,$$
$$H_1 = 0\,, \qquad H_4 = 5647.1\,,$$
$$H_2 = 42.645\,, \qquad H_5 = 85861\,.$$

For the lowest energy eigenvalues of the real proton we now get in the approximations of 0, 1, and 2 pions the values 0, -4.140, and -5.155. From now on we restrict ourselves to the two pion approximation. With $E = -5.155$ we solve (13) for $n = 2$. This gives

$$\psi_2 = (\alpha_0 + \alpha_1 H + \alpha_2 H^2)\Phi\,, \tag{22}$$

with

$$\alpha_0 = 0.5295\,, \quad \alpha_1 = -0.12498\,, \quad \alpha_2 = 0.005388\,. \tag{23}$$

Now we may calculate the probabilities for no, one, and two meson states. No meson state:

$$|(\psi_2, p\uparrow)|^2 = (\alpha_0 + \alpha_2 H_2)^2 = 0.57655\,. \tag{24}$$

One meson states: $\sum_n |(\psi_2, N\pi^{I_3}_{J_3}(n))|^2$ gives for $N\pi(n)$:

$$\mathrm{p}\uparrow\pi^0_0\,(\mathrm{n}):\ 0.04053\,,$$
$$\mathrm{p}\downarrow\pi^0_+\,(\mathrm{n}):\ 0.08106\,,$$
$$\mathrm{n}\uparrow\pi^+_0\,(\mathrm{n}):\ 0.08106\,,$$
$$\mathrm{n}\downarrow\pi^+_+\,(\mathrm{n}):\ 0.16212\,.$$

Thus the total probability for one meson states amounts to

$$\sum_{I_3, J_3}\sum_n |(\psi_2, N\pi^{I_3}_{J_3}(n))|^2 = 0.36478\,. \tag{25}$$

The probability for two meson states

$$\sum_{n,n'} |(\psi_2, N\pi^{I_3}_{J_3}(n)\pi^{I'_3}_{J'_3}(n'))|^2$$

expressed in units of $(\alpha_2 H_2/9)^2$ gives for

$$\begin{aligned}
&p\uparrow \pi_0^0\,\pi_0^0 : \ 2\,, \qquad \pi_0^+\,\pi_0^- : \ 4\,, \qquad \pi_+^+\,\pi_-^- : \ 16\,, \qquad \pi_+^0\,\pi_-^0 : \ 4\,,\\
&p\downarrow \pi_+^+\,\pi_0^- : \ 8\,, \qquad \pi_+^-\,\pi_0^+ : \ 8\,,\\
&n\uparrow \pi_+^+\,\pi_-^0 : \ 8\,, \qquad \pi_-^+\,\pi_+^0 : \ 8\,,\\
&n\downarrow \pi_+^+\,\pi_0^0 : \ 16\,, \qquad \pi_0^+\,\pi_+^0 : \ 1\,.
\end{aligned}$$

As $(\alpha_2 H_2/9)^2 = 0.000\,651\,9$ we get for the total probability of two meson states

$$\sum_{I_3 J_3 I_3' J_3' n n'} |(\psi_2,\ N\pi_{J_3}^{I_3}(n)\pi_{J_3'}^{I_3'}(n'))|^2 = 90(\alpha_2 H_2/9)^2 = 0.058\,67\,. \tag{26}$$

Now for r_1 and r_2 follows

$$r_1 = 0.451\,0\,, \qquad r_2 = 0.634\,0\,. \tag{27}$$

Therefore the renormalized coupling constant f is given by

$$f^2 = (r_2 f_0)^2 = 0.402\,0 \cdot 0.22 = 0.088\,44\,. \tag{28}$$

Thus the value of f^2 lies within the experimental errors. The mean meson number becomes

$$N_\pi = 0 \cdot 0.576\,55 + 1 \cdot 0.364\,78 + 2 \cdot 0.058\,67 = 0.482\,12\,. \tag{29}$$

Let the P_{IJ}'s be the probabilities for the pion cloud to stay in a state with isospin I and angular momentum J. Then the no meson state with 0.576 55 contributes only to P_{00}, the one meson state with 0.364 78 contributes only to P_{11}. The two meson state with $18\,(\alpha_2 H_2/9)^2 = 0.011\,73$ contributes to P_{00} and with $72\,(\alpha_2 H_2/9)^2 = 0.046\,94$ to P_{11}.

Thus we have

$$P_{00} = 0.588\,28\,, \qquad P_{01} = P_{10} = 0\,, \qquad P_{11} = 0.411\,72\,. \tag{30}$$

4. – The electromagnetic properties.

To study the electromagnetic properties of the model we start with the charge density $\varrho = \varrho_N + \varrho_\pi$, where

$$\varrho_N = \frac{\varepsilon}{2}(1 + \tau_3)\, U(\boldsymbol{x})\,, \qquad \varrho_\pi = -i\varepsilon\,(\pi_+\varphi_+ - \pi_-\varphi_-)\,. \tag{31}$$

From

$$\dot{Q}_\Omega = i[H, Q_\Omega]_- ,$$

where

$$Q_\Omega = \int_\Omega \mathrm{d}^3x\, \varrho ,$$

we get for the density $\boldsymbol{j}$ of the charge current

$$\boldsymbol{j} = \boldsymbol{j}_\pi + \boldsymbol{s} .$$

Here

$$\boldsymbol{j}_\pi = -i\varepsilon(\varphi_+ \boldsymbol{\nabla}\varphi_- - \varphi_- \boldsymbol{\nabla}\varphi_+) \tag{32}$$

is the meson current density. $\boldsymbol{s}$ is the gauge current with the property

$$\operatorname{div} \boldsymbol{s} = -i\varepsilon\sqrt{2\pi} f_0 \Big\{(\tau_+\varphi_+ + \tau_-\varphi_-)(\boldsymbol{\sigma}\boldsymbol{\nabla})\, U(\boldsymbol{x}) + \\ + U(\boldsymbol{x}) \int \mathrm{d}^3x' (\boldsymbol{\sigma}\boldsymbol{\nabla}')(\tau_-\varphi'_- + \tau_+\varphi'_+)\Big\} . \tag{33}$$

This means, we may add to $\boldsymbol{s}$ an arbitrary current the divergence of which vanishes. Therefore the contribution of $\boldsymbol{s}$ to the electromagnetic properties cannot be deduced from the static model but must be found from a comparison with a point nucleon. We may assume that $\boldsymbol{s}$ does not contribute essentially to the electromagnetic properties and we may restrict ourselves to the meson current only. For the mean square radius of the meson charge we find

$$\overline{r_\pi^2} = \int \mathrm{d}^3x\, r^2(\psi_2, \varrho_\pi \psi_2) = \\ = \frac{2}{\pi} f_0^2 \int_0^\infty \mathrm{d}k\, \frac{k^2}{\omega_k} \exp[-\alpha k^2] \Big\{(5\alpha k^2 - 1 - \alpha^2 k^4)[(\alpha_1 + \alpha_2\omega_k)^2 + 8\alpha_2^2 I_2] + \\ + [k^2 - (5 - 2\alpha k^2)\omega_k^2] \frac{k^2}{\omega_k^3} \alpha_2 (\alpha_1 + \alpha_2\omega_k)\Big\} = 0.021896 . \tag{34}$$

This means

$$\sqrt{\overline{r_\pi^2}} = \left\{\int \mathrm{d}^3x \langle r^2 \varrho_\pi\rangle \Big/ \int \mathrm{d}^3x \langle \varrho_\pi\rangle\right\}^{\frac{1}{2}} = (0.021396/0.27448)^{\frac{1}{2}} = 0.28244 ,$$

or

$$\sqrt{\overline{r^2}} = 0.4036 \text{ fermi} . \qquad (1 \text{ fermi} = 10^{-13} \text{ cm}). \tag{35}$$

For the nucleon we get

$$(36)\qquad \begin{cases} \overline{r^2_{\mathcal{N}}} = \int \mathrm{d}^3x\, r^2\, U(\boldsymbol{x}) = 4\pi(2\pi\alpha)^{-\frac{3}{2}} \int_0^\infty \mathrm{d}r\, r^4 \exp\left[-r^2/2\alpha\right] = 3\alpha = 0.147\ , \\ \sqrt{\overline{r^2_{\mathcal{N}}}} = 0.3834\,(\hbar/M_\pi c) = 0.548 \text{ fermi}\ , \end{cases}$$

if we put

$$M_\pi = \tfrac{1}{3}\left(M(\pi^+) + M(\pi^0) + M(\pi^-)\right) = 270.3\, M_e\ .$$

The mean squared radius (m.s.r.) of the whole charge distribution now becomes

$$(37)\qquad \begin{cases} \overline{r^2} = \dfrac{1}{\varepsilon}\int \mathrm{d}^3x\, r^2\, \varrho = \dfrac{1}{\varepsilon}\int \mathrm{d}^3x r^2 \langle \varrho_\pi + \varrho_{\mathcal{N}} \rangle = \\ \qquad\qquad = 0.021896 + 0.106652 = 0.128548\ , \\ \sqrt{\overline{r^2}} = 0.5124 \text{ fermi}\ . \end{cases}$$

According to this model the m.s.r. of the meson cloud is somewhat smaller than the m.s.r. of the bare proton. We compare this result with the electron-proton scattering experiments of HOFSTADTER *et al.* [7]. We assume a uniform Gaussian distribution for the charge density and the density of the magnetic moment. Then the differential cross-section for electron-proton scattering should be

$$(38)\qquad \frac{\mathrm{d}\sigma}{\mathrm{d}\omega} = \left(\frac{\varepsilon}{2E_0}\right)^2 \frac{\cos^2\vartheta/2}{\sin^4\vartheta/2}\left(1 + \frac{2E_0}{M_p}\sin^2\frac{\vartheta}{2}\right)^{-1} \cdot \left\{1 + \left(\frac{p}{2M_p}\right)^2\left[2\mu_p \operatorname{tg}^2\frac{\vartheta}{2} + (\mu_p - 1)^2\right]\right\} \exp\left[-\tfrac{1}{3}\overline{r^2}p^2\right],$$

$p = 2E_0(1+(E_0/M_p)\sin^2\vartheta/2)^{-1}\sin^2\vartheta/2$. E_0 is the total energy of the electron, ϑ its scattering angle, M_p is the proton mass, $\mu_p = 2.7928$ is the total magnetic moment of the proton expressed in nuclear magnetons. For $E_0 = 188$ MeV the experiments are compatible with

$$\sqrt{\overline{r^2}} = (0.34 \pm 0.30) \text{ fermi}\ .$$

For $E_0 = 550$ MeV we get $\sqrt{\overline{r^2}} = (0.66 \pm 0.12)$ fermi. As for higher energies the magnetic moment distribution plays the dominant role, these results are not too conclusive for the charge distribution. The theoretical distribution is not too bad, though it seems to be somewhat too concentrated around the origin.

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[7] R. HOFSTADTER, F. BUMILLER and M. R. YEARIAN: *Rev. Mod. Phys.*, **30**, 482 (1956).

Therefore also the virtual mesons get a higher energy so that they are more inside than outside of the nucleon.

For the pion part of the magnetic moment we get

$$\mu_\pi = \frac{1}{2}\int \mathrm{d}^3x \langle x_1 j_{\pi 1} - x_2 j_{\pi 1}\rangle = \frac{\varepsilon}{2}\sum_{n,m}\frac{m}{\omega_n}\langle a^{n\dagger}_{-,m}a^n_{+,m} - a^{n\dagger}_{-,m}a^n_{-,m} + \tag{39}$$

$$+ (-1)^m(a^{n\dagger}_{-,m}a^{n\dagger}_{+,-m} - a^n_{+,m}a^n_{-,-m})\rangle\ .$$

If we use our two pion approximation we get

$$\langle\mu_\pi\rangle/\mu_k = 4M_p\{[2(\alpha_0 + \alpha_2 H_2)\alpha_2 + \alpha_1^2]I_1 + (2\alpha_1 I_2 + \alpha_2 I_3)\alpha_2\} = 0.419\,7\ . \tag{40}$$

The contributions of the proton and the neutron are

$$\langle \tfrac{1}{2}(1 \pm \tau_3)\sigma_3\rangle = \tfrac{1}{2}(r_1 \pm r_2) = \tfrac{1}{2}(0.451\,05 \pm 0.634\,03) = 0.542\,54\,\mu_k$$

for the proton and $-0.091\,49\,\mu_k$ for the neutron. Thus we get

$$\begin{cases} \mu_p = (0.542\,54 + 0.419\,73)\mu_k = 0.962\,27\,\mu_k\ , \\ \mu_n = (-0.091\,49 - 0.419\,73)\mu_k = -0.511\,22\,\mu_k , \end{cases} \tag{41}$$

and

$$\mu_p + \mu_n = r_1\mu_k = 0.451\,05\,\mu_k\ . \tag{42}$$

These results would mean that either the gauge current $\boldsymbol{s}$ should contribute about $1.5\mu_k$ to the magnetic moments of the nucleons or also that the pion cloud is too small so that the pions get too high energy values and therefore they give too small a contribution to the magnetic moments.

On the other hand the experimental value

$$\mu_p + \mu_n = (2.792\,8 - 1.931\,1)\mu_k = 0.879\,7\,\mu_k$$

would mean $r_1 = 0.879\,7$, which value is surely too high, so that we would expect an essential contribution of heavier mesons, which change the magnetic moments of the proton and the neutron in a non-symmetric way.

5. – The strong sum rules.

If we now substitute our nucleon distribution into formula (20) we get $r_1 = (0.333 \pm 0.071)$.

The errors arise in fact only from the errors of $f^2 = 0.083 \pm 0.05$. On the

other hand in (20) it makes nearly no difference, if we identify the system of the static model with the centre of mass system or with the laboratory system of the interacting pion and nucleon (r_1 only changes by 0.01).

MIYAZAWA ([8]) gives the sum rules

$$f_0^2 - f^2 = \frac{1}{8\pi^2}\int \frac{\mathrm{d}k}{\omega_k}\{\sigma^-(k) + \sigma^+(k)\} = 0.11\,, \tag{43}$$

and

$$\frac{1}{2}(1 - r_1)f_0^2 = \frac{1}{8\pi^2}\int \frac{\mathrm{d}k}{\omega_k}\sigma^+(k) = 0.068\,. \tag{44}$$

(43) and (44) imply the relation

$$(1 - r_2^2)/(1 - r_1) = (0.11/2\cdot 0.068) = 0.81$$

or

$$r_2^2 - 0.81\, r_1 = 0.19\,. \tag{45}$$

It has been shown by FUBINI and THIRRING ([6]) that the strong sum rule

$$1 + 2r_1 - 3r_2 \geqslant 0 \tag{46}$$

must hold. But (45) is in contradiction to the relation (46). Therefore (45) cannot hold exactly. If we substitute our values

$$r_1 = 0.451\,, \qquad r_2 = 0.634\,, \qquad f_0^2 = 0.22\,, \qquad f^2 = 0.088\,,$$

we get

$$f_0^2 \geqslant f^2 = (1 - r_2^2)f_0^2 = 0.134\,,$$

$$\tfrac{1}{2}(1 - r_1)f_0^2 = 0.060\,4\,,$$

$$r_2^2 - 0.81\, r_1 = 0.037\,1\,,$$

$$1 + 2r_1 - 3r_2 = 0\,.$$

The results mean that our value for $2r_1 - 3r_2$ is the highest one which is compatible with the inequality (46). But also the values of (43) and (44) do not deviate too much from our values of $(1 - r_2^2)f_0^2$ and $\frac{1}{2}(1 - r_1)f_0^2$. Therefore it seems that the errors of (43) and (44) cannot be very large.

Our value for the mean pion number $N_\pi = 0.48$ does not fit the sum rule

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([8]) H. MIYAZAWA: *Phys. Rev.*, **101**, 1564 (1956).

of FUBINI and THIRRING

$$(47) \qquad N_\pi = \frac{3}{\pi}\int_0^\infty \frac{v^2(k)k^4\,\mathrm{d}k}{\omega_k}\left\{\frac{f^2}{\omega_k^2} + \frac{1}{3\pi}\int_1^\infty \frac{\mathrm{d}\omega_p[\beta_{11}(\omega_p) + 4\beta_{13}(\omega_p) + 4\beta_{33}(\omega_p)]}{p^3 r^2(p)(\omega_k + \omega_p)^2}\right\}.$$

If we substitute $f^2 = 0.083 \pm 0.05$ and $v^2(k) = \exp[-\alpha k^2]$ we get $N_\pi = 1.07 \pm 0.04$ in contradiction to (29). Perhaps this also is an indication for a too small source, because if we take $\alpha = 0.07$ instead of 0.049 we get from (47) $N_\pi = 0.69 \pm 0.03$ and $\sqrt{\overline{r_N^2}} = 0.65$ fermi instead of 0.55 fermi.

6. – Conclusion.

Our results compared with the strong coupling limit are as follows:

		strong coupling
P_{00}	0.588 3	0.25
$P_{01} = P_{10}$	0	0
P_{11}	0.411 7	0.75
r_1	0.451 0	0
r_2	0.634 0	0.333 3

The values obtained—though only in the two pion approximation—seem to be far enough from the strong coupling limit, that we may expect a realistic static model also with weakly coupled pions.

* * *

I wish to thank Prof. Dr. W. THIRRING for the inspiration of this work.

RIASSUNTO (*)

Il metodo degli impulsi proposto da F. HALPERN (1,2) viene applicato alla descrizione dello stato reale del nucleone nell'approssimazione del modello a due mesoni. Si è usato il modello statico di Chew e Low per una sorgente estesa. Sembra possibile ottenere un modello ad accoppiamento debole che dà un buon valore di f^2 e che si accorda abbastanza bene con le regole di somma per le ampiezze di scattering pione-nucleone.

(*) *Traduzione a cura della Redazione.*

Relativistic Deuteron Wave Function - II (*).

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Summary. -- Carrying on our research on the deuteron relativistic wave function, we have applied Bethe and Salpeter's relativistic wave equation to the bound system of two particles with spin $\frac{1}{2}$. After some very slight approximations, the solution of the infinite system of coupled integral equations enables us to find the explicit structure of the wave matrices corresponding to the S and D states. The percentage of D state obtained is in good agreement with the experimental results.

1. – Introduction.

We have already studied the problem of the determination of a relativistic deuteron wave function (1). In this paper, we investigate the case of two Dirac nucleons with the Bethe and Salpeter equation formalism (2). The expansion of the wave function in hyperspherical harmonics has been first introduced in the nucleon-nucleon scattering problem (3). It was used in our

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(1) M. Gourdin and J. Tran Thanh Van: *Nuovo Cimento*, **14**, 1051 (1959).

(2) E. E. Salpeter and H. A. Bethe: *Phys. Rev.*, **84**, 1232 (1951).

(3) M. Gourdin: *Ann. de Phys.* **4**, 595 (1959); *Nuovo Cimento*, **7**, 338 (1958).

first paper for a spinless deuteron. We employ here the PS (PS) meson theory with an interaction invariant under rotations.

We obtain an infinite coupled system of radial equations for the 16 components of the wave function. In the first section, we develop some calculations in order to obtain the equations corresponding to the $J=1$ bound neutron-proton state. Because of parity conservation, we can use selection rules to simplify the equations.

In the second section, we analyse the structure of the integral equations. There appear three quantum numbers L, m, n due to the expansion in hyperspherical harmonics, of the three angles θ, φ, β, corresponding to the four-vector $\boldsymbol{p}$. The first one L is the orbital angular momentum; the second m its projection over Oz. The last one n has no direct physical interpretation; its parity is conserved and it is responsible for the coupling between integral equations. But it is very useful to note that this coupling is extremely weak; practically the lowest value for n is important and we can reduce the determination of the wave functions to the resolution of one system of two homogeneous coupled integral equations. The splitting of the wave matrix into S and D parts becomes trivial.

In the third section, we use various methods to solve these equations. The eigenvalue of the integral kernel possesses only discrete values and the lower one is proportional to the coupling constant of nuclear forces. The computations performed in the ladder approximation lead to values of $g^2/4\pi$ less than 8 (experimental value $\simeq 15$). The inclusion in the interaction kernel of fourth-order terms appears as well necessary as prohibitive from a practical point of view.

In the last section, we present a simplified method to obtain a relativistic wave function. The relativistic corrections are divided into two types. The kinematical corrections and the properties of symmetry correctly describe the spin structure of the Deuteron which is represented by an entirely known matrix. The relativistic corrections about the dynamics of the system are partially included in a scalar function which is equivalent, in a non relativistic limit, to Hulthén's wave function. We hope this solution, essentially phenomenological, is reasonable and sufficiently simple to permit calculation. The corresponding percentage of D state in the deuteron is 4% in agreement with experimental results.

2. – Integral equations for the deuteron wave function.

2·1. *System of two bound nucleons with spin* $\frac{1}{2}$. – The nucleon is represented by a four-component spinor and the two-nucleon wave-function $\Phi(\mathbf{p})$ is a 4×4 matrix, the « direct product » in a loose sense of the two spinors for the two particles.

This matrix satisfies the Bethe and Salpeter equation

$$(1) \qquad \Phi(\mathbf{p}) = \alpha \frac{(\gamma^{(1)} p_1 + iM)(\gamma^{(2)} p_2 + iM)}{(p_1^2 + M^2)(p_2^2 + M^2)} \gamma_5^{(1)} \gamma_5^{(2)} \int W(\mathbf{p}, \mathbf{p}') \Phi(\mathbf{p}') \, d\mathbf{p}'$$

if we consider a PS (PS) coupling.

The four-moments $\mathbf{p}_1$, $\mathbf{p}_2$ are related to the moment of the center of mass $\mathbf{P}$ and the relative moment $\mathbf{p}$ through the equations:

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 , \qquad 2\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 ,$$

or

$$\mathbf{p}_1 = \frac{\mathbf{P}}{2} + \mathbf{p} , \qquad \mathbf{p}_2 = \frac{\mathbf{P}}{2} - \mathbf{p} .$$

The matrices $\gamma_\mu^{(i)}$ relative to the particle (i) act on $\Phi(p)$ from the left if $i = 1$ and, after transposition, to the right if $i = 2$ [4],

$$\gamma_\mu^{(1)} \Phi(\mathbf{p}) = \gamma_\mu \Phi(\mathbf{p}) ,$$
$$\gamma_\mu^{(2)} \Phi(\mathbf{p}) = \Phi(\mathbf{p}) \gamma_\mu^T ,$$

γ_μ being an usual Dirac matrix and γ_μ^T, its transpose.

Let us introduce the B matrix:

$$\gamma_\mu^T = B^{-1} \lambda_\mu B$$

and the wave function

$$\Psi(\mathbf{p}) = \Phi(\mathbf{p}) B^{-1} ,$$

then we get the simple relations:

$$\gamma_\mu^{(1)} \Psi = \gamma_\mu \Psi ,$$
$$\gamma_\mu^{(2)} \Psi = \Psi \gamma_\mu .$$

The new wave matrix $\Psi(\mathbf{p})$ satisfies the integral equation:

$$(2) \qquad \Psi(\mathbf{p}) = \frac{\alpha}{(p^2 + M^2 + (P^2/4)) - p_4^2 P^2} \cdot \int W(\mathbf{p}, \mathbf{p}') \, | (\gamma \mathbf{p}_1 + iM) \gamma_5 \Psi(\mathbf{p}') \gamma_5 (\gamma \mathbf{p}_2 + iM) | \, d\mathbf{p}' .$$

[4] L. DE BROGLIE: *Théorie générale des particules à spin (Méthode de fusion)*.

Let us write $\Psi(\mathbf{p})$ in a reduced form having four elements which are 2×2 matrices

$$\Psi(\boldsymbol{p}) = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix}.$$

Let us now express the 2×2 matrices in terms of the unit matrix and the Pauli matrices

$$\Psi_{11}(\mathbf{p}) = S_1(\mathbf{p}) I + \boldsymbol{\sigma} \boldsymbol{V}_1(\mathbf{p}),$$

$$\Psi_{22}(\mathbf{p}) = S_2(\mathbf{p}) I + \boldsymbol{\sigma} \boldsymbol{V}_2(\mathbf{p}),$$

$$\Psi_{12}(\mathbf{p}) = \frac{B(\mathbf{p}) + C(\mathbf{p})}{2} I + \boldsymbol{\sigma} \frac{\boldsymbol{F}(\mathbf{p}) + \boldsymbol{G}(\mathbf{p})}{2},$$

$$\Psi_{21}(\mathbf{p}) = \frac{B(\mathbf{p}) - C(\mathbf{p})}{2} I + \boldsymbol{\sigma} \frac{\boldsymbol{F}(\mathbf{p}) - \boldsymbol{G}(\mathbf{p})}{2}.$$

If we substitute these expressions for the reduced elements of $\Psi(\mathbf{p})$ in eq. (2) we get a system of coupled homogeneous integral equations which has already been written down for the scattering problem ([3]).

Using the same method as for the problem of two particles without spin ([1]) we expand the scalar functions $S_1(\mathbf{p})$, $S_2(\mathbf{p})$, $B(\mathbf{p})$, $C(\mathbf{p})$ in hyperspherical harmonics and the vector functions in vectorial hyperspherical harmonics. We assume, as was done before, that the interaction kernel $W(\mathbf{p}, \mathbf{p}')$ depends only on the lenghts of the four vectors p, p' and on the angle (H) between them.

This assumption allows us to integrate over the angles θ and φ, the hyperspherical harmonics, both scalar and vectorial, being orthonormalized. When performing the integration over the third angle β we are led to introduce a lot of auxiliary functions which are the elements of a 16×16 matrix which we call $K(p; J, n, n')$. This matrix will eventually act on the 16 component vector $\Psi(\mathbf{p})$ which was written as a 4×4 matrix itself. In the case of nucleons without spin, K was shown to be simply the function $E_0(p; 1, n, n')$. We order the 16 components of $\Psi(\mathbf{p})$ in an arbitrary way; the non vanishing elements of K indicate between what radial components of $\Psi(\mathbf{p})$ a coupling does exist. We have established in a former work on the nucleon-nucleon scattering ([3]) the following results:

a) The K matrix is expressed in terms of four 8×8 sub-matrices. The two non-diagonal ones are zero, the two diagonal ones refer to a definite value of the parity, the total angular momentum J being a constant of the motion.

b) The sub matrix of parity $(-1)^J$ is in turn expressed in the same way, in terms of two 4×4 sub-matrices lying along the diagonal referring respectively to singlet ($J = L$, $m = 0$) and triplet ($J = L$, $m = \pm 1$) states. m is the azimuthal quantum number.

2·2. *The deuteron case*: $J=1$ *state*. – For a deuteron in the ground state, $J=1$ and parity is $+1$. The wave function will, *a priori*, have 8 components and the K matrix will be the sub-matrix 8×8 of parity $+1$.

Let us write in the following order the radial components $A(p; JLmn)$ of $\Psi(\mathbf{p})$, omitting the index J equal to 1 and the index L when L is equal to 1

$$(3)\qquad \begin{cases} V_1(p;2mn)\,; & V_1(p;0mn)\,; & V_2(p;2mn)\,; & V_2(p;0mn) \\ B(p;mn)\,; & G(p;mn)\,; & C(p;mn)\,; & F(p;mn)\,. \end{cases}$$

On the other hand, the matrix $K(p;nn')$ is an integral over the angle β of another matrix $R(p,\beta)$:

$$(4)\qquad K_{\lambda\mu}(p;n,n') = \int_0^{\pi} \frac{\mathscr{C}_{l}^{n}(\beta)\, R_{\lambda\mu}(p,\beta)\, \mathscr{C}_{l'}^{n'}(\beta)}{\left(p^2+M^2+(P^2/4)\right)^2 - p^2P^2\cos^2\beta}\sin^2\beta\,\mathrm{d}\beta\,.$$

TABLE I. – *R-matrix*.

$R_{11}=-\dfrac{p^2\sin^2\beta}{3}$	$R_{12}=2\sqrt{2}\,R_{11}$	$R_{13}=-\left[p^2\cos^2\beta+\left(2M-\dfrac{B}{2}\right)^2\right]$	$R_{14}=0$
$R_{15}=\sqrt{\dfrac{2}{3}}\sin\beta\left(2M-\dfrac{B}{2}\right)$	$R_{16}=\dfrac{1}{\sqrt{2}}R_{15}$	$R_{17}=i\sqrt{\dfrac{2}{3}}\,p^2\sin\beta\cos\beta$	$R_{18}=\dfrac{1}{\sqrt{2}}R_{17}$
$R_{21}=R_{12}$	$R_{22}=-R_{11}$	$R_{23}=0$	$R_{24}=R_{13}$
$R_{25}=R_{16}$	$R_{26}=-R_{15}$	$R_{27}=R_{18}$	$R_{28}=-R_{17}$
$R_{31}=-\left[p^2\cos^2\beta+\dfrac{B^2}{4}\right]$	$R_{32}=0$	$R_{33}=R_{11}$	$R_{34}=R_{12}$
$R_{35}=-\sqrt{\dfrac{2}{3}}\,p\sin\beta\,\dfrac{B}{2}$	$R_{36}=-\dfrac{1}{\sqrt{2}}R_{35}$	$R_{37}=R_{28}$	$R_{38}=-R_{27}$
$R_{41}=0$	$R_{42}=R_{31}$	$R_{43}=R_{21}$	$R_{44}=+R_{22}$
$R_{45}=-R_{25}$	$R_{46}=R_{35}$	$R_{47}=R_{28}$	$R_{48}=R_{17}$
$R_{51}=-2R_{35}$	$R_{52}=2R_{36}$	$R_{53}=-2R_{15}$	$R_{54}=-2R_{16}$
$R_{55}=p^2-\gamma^2$	$R_{56}=0$	$R_{57}=-2ipM\cos\beta$	$R_{58}=0$
$R_{61}=R_{52}$	$R_{62}=-R_{51}$	$R_{63}=-R_{54}$	$R_{64}=R_{53}$
$R_{65}=0$	$R_{66}=-R_{55}$	$R_{67}=0$	$R_{68}=-R_{57}$
$R_{71}=-2R_{17}$	$R_{72}=-2R_{18}$	$R_{73}=2R_{17}$	$R_{74}=2R_{18}$
$R_{75}=-R_{57}$	$R_{76}=0$	$R_{77}=-p^2\cos 2\beta+\gamma^2$	$R_{78}=0$
$R_{81}=-2R_{18}$	$R_{82}=2R_{28}$	$R_{83}=-2R_{28}$	$R_{84}=2R_{27}$
$R_{85}=0$	$R_{86}=R_{57}$	$R_{87}=0$	$R_{88}=-R_{77}$

The indices l, l' are related respectively to the components λ, μ. With the above choice (3) for numbering the radial components $A_\lambda(p; l, m, n)$ we have

$$(5) \qquad \begin{cases} l = 0 & \text{for } \lambda = 2, 4, \\ l = 1 & \text{for } \lambda = 5, 6, 7, 8, \\ l = 2 & \text{for } \lambda = 1, 3 . \end{cases}$$

The matrix R is given in the Table I.

The selection rules due to the parity of the Gegenbauer polynomials show that certain elements of K will be zero after integration over β. Thus, the components A_1, A_2, A_3, A_4, A_5, A_6 will be real while the components A_7, A_9 will be imaginary; we have $n+n' = l+l'+2q$ for the former and $n+n' = l+l'+2q+1$ for the latter.

Then, we obtain for the radial components $A_\lambda(p; m, n)$ an infinite system of integral equations coupled by the indices n and λ,

$$(6) \qquad A_\lambda(p; m\,n) = \alpha \sum_{n'} \sum_{\mu} K_{\lambda\mu}(p; n\,n') \int_0^\infty \Delta_{n'}(p, p') A_\mu(p'; m\,n') p'^3 \,\mathrm{d}p' .$$

3. – Approximate resolution of the system of integral equations.

Using the same approximation methods as in the preceding article [1] we propose to find a solution to the system of integral equations (6). We saw that these equations were very weakly coupled with respect to the index n and that the wave functions fell off very fast as n increased. The function $W(\mathbf{p}, \mathbf{p}')$ which is the scalar part of the interaction without the Dirac matrices corresponds to the same function $W(\mathbf{p}, \mathbf{p}')$ introduced for the simplified scalar problem. Thus it turns out that the function $\Delta_n(p, p')$ are the same in the two cases. Actually we have employed the ladder approximation but the arguments for the convergence of series with the index n remain valid for physically reasonable interactions. The falling off of $\Delta_n(p, p')$ with n is very fast, we shall decouple the equations and conserve for a given component only the term of the smallest order in n.

3·1. *Terms $n = 0$ and $n' = 0$.* – The relation connecting l, l', n, n' and q shows that, in this case, only two components intervene: $V_1(p; 0\,0\,0)$ and $V_2(p; 0\,0\,0)$ which we call $V_1^-(p)$ and $V_2^-(p)$. The system of integral equa-

tions (6) is then reduced to the two coupled equations:

$$(7)\quad \begin{cases} V_1^-(p) = \alpha\left[K_{22}(p)\displaystyle\int_0^\infty \Delta_0(p, p')\, V_1^-(p') p'^3\, \mathrm{d}p' + K_{24}(p)\int_0^\infty \Delta_0(p, p')\, V_2^-(p') p'^3\, \mathrm{d}p'\right], \\ V_2^-(p) = \alpha\left[K_{42}(p)\displaystyle\int_0^\infty \Delta_0(p, p')\, V_1^-(p') p'^3\, \mathrm{d}p' + K_{44}(p)\int_0^\infty \Delta_0(p, p')\, V_2^-(p') p'^3\, \mathrm{d}p'\right]. \end{cases}$$

Henceforth, by convention, $K_{\lambda\mu}(p)$ is the function $K_{\lambda\mu}(p; n, n')$ corresponding to the minimum values of n and n' for which it is not zero. Their calculation becomes that of the functions $E_0(p; l\,0\,0)$,

$$K_{22}(p) = K_{44}(p) = \frac{p^2}{4}\left(1 - \frac{h(p)}{3}\right) E_0(p),$$

$$K_{24}(p) = -\left[\frac{p^2}{4}(1 + h(p)) + 4M^2\right] E_0(p),$$

$$K_{42}(p) = -\left[\frac{p^2}{4}(1 + h(p)) + \left(\frac{\gamma^2}{4M}\right)^2\right] E_0(p).$$

Recall

$$E_0(p) = \frac{2}{(p^2+\gamma^2)\{(p^2+\gamma^2)^2 + 4p^2M^2\}^{\frac{1}{2}}}$$

$$h(p) = \frac{(p^2+\gamma^2) - \{(p^2+\gamma^2)^2 + 4p^2M^2\}^{\frac{1}{2}}}{(p^2+\gamma^2) + \{(p^2+\gamma^2)^2 + 4p^2M^2\}^{\frac{1}{2}}}$$

and the binding energy of the deuteron $-B$ appears through the parameter γ thus defined

$$\gamma^2 = MB.$$

To facilitate calculation we define two auxiliary functions $I_1(p)$ and $I_2(p)$

$$(8)\quad I_{1,2}(p) = \int_0^\infty \Delta_0(p, p')\, V_{1,2}^-(p') p'^3\, \mathrm{d}p'$$

permitting us to write the system (7) as

$$(9)\quad \begin{cases} V_1^-(p) = \alpha[K_{22}(p)\, I_1(p) + K_{24}(p)\, I_2(p)], \\ V_2^-(p) = \alpha[K_{42}(p)\, I_1(p) + K_{22}(p)\, I_2(p)]. \end{cases}$$

3·2. *Terms* $n=1$, $n'=0$. – At this order of the approximation, two new components of the wave matrix $B(p;\,1\;m\;1)$ and $G(p;\,1\;m\;1)$ appear. We call them $B(p)$ and $G(p)$ and express them with the help of I_1 and I_2.

$$\left\{\begin{aligned} B(p) &= \alpha[K_{52}(p)\,I_1(p) + K_{54}(p)\,I_2(p)]\,,\\ G(p) &= \alpha[K_{62}(p)\,I_1(p) + K_{64}(p)\,I_2(p)]\,. \end{aligned}\right. \tag{10}$$

The four new functions $K_{\lambda\mu}(p)$ are then given in terms of $E_1(p)$ by

$$K_{52}(p) = \left(\frac{\gamma}{2M}\right)^2 2pME_1(p)\,, \qquad K_{51}(p) = -2pME_1\,,$$

$$K_{62}(p) = -\left(\frac{\gamma}{2M}\right)^2 2\sqrt{2}\,pME_1(p)\,, \qquad K_{61}(p) = -2\sqrt{2}\,pME_1(p)\,,$$

where

$$E_1(p) = \left(1-\frac{h(p)}{3}\right)E_0(p)\,.$$

We can invert the equations (9)

$$I_1 = \frac{1}{\alpha W}\,[K_{22}V_1^- - K_{21}V_2^-]\,,$$

$$I_2 = \frac{1}{\alpha W}\,[-K_{42}V_1^- + K_{41}V_2^-]$$

where we put

$$W(p) = K_{22}K_{41} - K_{21}K_{42}$$

and see that $B(p)$ and $G(p)$ are, at this point in the calculation, simple linear combination of $V_1^-(p)$ and $V_2^-(p)$, the coefficients being expressed in terms of the functions $K_{\lambda\mu}(p)$.

Finally one should notice that, the binding energy of the deuteron being very small ($B=2.23$ MeV), the number $(\gamma/2M)^2$ is much less than unity ($\simeq 1/2\,000$); $K_{52}(p)$ and $K_{62}(p)$ are thus negligible before $K_{54}(p)$ and $K_{61}(p)$. Doing this, we find a simple relation between B and G

$$G(p) = \sqrt{2}\,B(p) \tag{11}$$

which is certainly valid to a very good approximation.

3·3. *Terms* $n=0$, $n'=1$. – The equations (7) are now modified by the addition of a coupling term between $V_1^-(p)$, $V_2^-(p)$ and $B(p)$, $G(p)$. We thus calculate a first correction to the solutions of the equations (7).

If we introduce the auxiliary functions $I_3(p)$ and $I_4(p)$

$$I_3(p) = \int_0^\infty A_1(p, p')\, B(p') p'^3 \, \mathrm{d}p' ,$$

$$I_4(p) = \int_0^\infty A_1(p, p')\, G(p') p'^3 \, \mathrm{d}p' ,$$

the system (9) becomes

$$\text{(12)} \qquad \begin{cases} V_1^-(p) = \alpha [K_{22} I_1 + K_{24} I_2 + K_{25} I_3 + K_{26} I_4] , \\ V_2^-(p) = \alpha [K_{42} I_1 + K_{44} I_2 + K_{45} I_3 + K_{46} I_4] , \end{cases}$$

with

$$K_{25}(p) = pME_1 , \qquad K_{26}(p) = -\sqrt{2}\, K_{25}(p) ,$$

$$K_{45}(p) = -\left(\frac{\gamma}{2M}\right)^2 K_{25}(p) , \qquad K_{46}(p) = \left(\frac{\gamma}{2M}\right)^2 K_{26}(p) .$$

The problem is then to solve the coupled systems (10) and (12). By eliminating $B(p)$ and $G(p)$ we obtain two coupled integral equations for $V_1^-(p)$ and $V_2^-(p)$ alone. The difference between this system and (8) will enable us to test our development in series with respect to n.

But, it is legitimate to neglect $(\gamma/2M)^2$ before unity. Relation (11) is then valid and the system (12) is equivalent to

$$\text{(13)} \qquad \begin{cases} V_1^-(p) = \alpha [K_{22} I_1 + K_{24} I_2 - K_{25} I_3] , \\ V_2^-(p) = \alpha [K_{42} I_1 + K_{44} I_2] . \end{cases}$$

The second equation (9) is not modified, so that, taking (8), (9), (11) into account, we get only one substitution,

$$K_{24}(p)\, A_0(p, p') \Rightarrow K_{24}(p)\, A_0(p, p') - K_{25}(p) \int_0^\infty A_1(p, p'')\, K_{51}(p'')\, A_0(p'', p') p''^3 \, \mathrm{d}p'' ,$$

which can be easily calculated in the plane (p, p') since all the elements are known.

3·4. *Terms* $n = 2$, $n' = 0$ (*real*). – In this case, the orbital angular momentum components of order 2 appear. They are $V_1(p; 2\,m\,2)$ and $V_2(p; 2\,m\,2)$,

which we call $V_1^+(p)$ and $V_2^+(p)$ respectively. They are easily known once $V_1^-(p)$ and $V_2^-(p)$ have been calculated,

$$(14) \qquad \begin{cases} V_1^+(p) = \alpha K_{12}(p) I_1(p) , \\ V_2^+(p) = \alpha K_{34}(p) I_2(p) . \end{cases}$$

The two functions $K_{12}(p)$ and $K_{34}(p)$ are equal

$$K_{12}(p) = K_{34}(p) = -\frac{\sqrt{5}}{3} p^2 \left[1 - \frac{h(p)}{2} + \frac{h^2(p)}{10} \right] E_0(p) .$$

3·5. *Terms* $n = 2$, $n' = 0$ (*imaginary*). – The two imaginary functions that we introduce here $C(p; 1\, m\, 2)$ and $F(p; 1\, m\, 2)$ are to be multiplied by the function $\mathscr{C}_1^2(\beta)$. It is easy to see that in configuration space, the same thing happens. At the limit of equal times, $\beta = \pi/2$ and $\mathscr{C}_1^2(\pi/2) = 0$. For that reason, we do not worry about these imaginary functions.

3·6. *Conclusion.* – Taking into account in each case only the lowest terms in n and n', we are led to the following programme:

a) Solve the system (7) to obtain $V_1^-(p)$, $V_2^-(p)$, I_1 and I_2;
b) Calculate $B(p)$ and $G(p)$ by equations (10);
c) Calculate $V_1^+(p)$ and $V_2^+(p)$ by equations (14).

4. – Separation into states $^3S_1$ and $^3D_1$.

The wave matrix $\Psi(\mathbf{p})$ in our particular case has therefore the form:

$$\Psi(\mathbf{p}) = \begin{vmatrix} \boldsymbol{\sigma} \cdot \boldsymbol{V}_1(\mathbf{p}) & \dfrac{B(p)}{2} + \dfrac{\boldsymbol{\sigma} \boldsymbol{G}(\mathbf{p})}{2} \\ \dfrac{B(\mathbf{p})}{2} - \dfrac{\boldsymbol{\sigma} \boldsymbol{G}(\mathbf{p})}{2} & \boldsymbol{\sigma} \cdot \boldsymbol{V}_2(\mathbf{p}) \end{vmatrix} .$$

In the state of total angular momentum $J = 1$, it becomes:

$$\begin{aligned} \boldsymbol{V}_{1,2}(\mathbf{p}) &= V_{1,2}^-(p)\, \mathscr{Y}_{101}^{m\,0}(\theta, \varphi, \beta) + V_{1,2}^+(p)\, \mathscr{Y}_{121}^{m\,0}(\theta, \varphi, \beta) , \\ B(\mathbf{p}) &= B(p)\, \mathscr{C}_1^1(\beta)\, \mathscr{Y}_1^m(\theta, \varphi) , \\ \boldsymbol{G}(\mathbf{p}) &= G(p) \cdot \mathscr{Y}_{111}^{m1}(\theta, \varphi, \beta) . \end{aligned}$$

The vectorial hyperspherical harmonics $\mathscr{Y}_{J\,l\,1}^{m\,n}(\theta, \varphi, \beta)$ may be obtained from

scalar hyperspherical harmonics by the relations:

$$\mathscr{Y}^{m,n}_{JJ1} = \frac{1}{\sqrt{J(J+1)}}\, \boldsymbol{L}\mathscr{Y}^{mn}_{J},$$

$$\mathscr{Y}^{m,n}_{J,J+1,1} = \frac{1}{\sqrt{(J+1)(2J+1)}}\, [(J+1)\,\boldsymbol{e}_p + i\,\boldsymbol{e}_p \times \boldsymbol{L}]\,\mathscr{Y}^{m,n}_{J},$$

$$\mathscr{Y}^{m,n}_{J,J-1,1} = \frac{1}{\sqrt{(J+1)(2J+1)}}\, [J\,\boldsymbol{e}_p - i\,\boldsymbol{e}_p \times \boldsymbol{L}]\,\mathscr{Y}^{m,n}_{J}.$$

We easily deduce the two relations

$$\mathscr{Y}^{mn}_{J} = (\boldsymbol{\sigma e}_p)\cdot\boldsymbol{\sigma}\left[\sqrt{\frac{J+1}{2J+1}}\,\mathscr{Y}^{m,n}_{J,J+1,1} + \sqrt{\frac{J}{2J+1}}\,\mathscr{Y}^{m,n}_{J,J-1,1}\right],$$

$$\boldsymbol{\sigma}\,\mathscr{Y}^{mn}_{JJ1} = (\boldsymbol{\sigma e}_p)\cdot\boldsymbol{\sigma}\left[\sqrt{\frac{J}{2J+1}}\,\mathscr{Y}^{mn}_{J,J+1,1} - \sqrt{\frac{J+1}{2J+1}}\,\mathscr{Y}^{mn}_{J,J-1,1}\right],$$

using the property $\boldsymbol{e}_p \boldsymbol{L} = 0$, $\boldsymbol{e}_p$ being the unitary vector in the direction $\boldsymbol{p}$.

Applying these equations for $J = 1$, the wave matrix becomes simply:

$$\Psi(\mathbf{p}) = M_S(p,\beta)\boldsymbol{\sigma}\mathscr{Y}^{m}_{101}(\theta,\varphi) + M_D(p,\beta)\boldsymbol{\sigma}\mathscr{Y}^{m}_{121}(\theta,\varphi).$$

Here $\mathscr{Y}^{m}_{JL1}$ are ordinary vector orthonormal spherical harmonics. The wave matrices for the S and D states are given by

$$(16)\qquad \begin{cases} M_S = \sqrt{\dfrac{2}{\pi}} \begin{vmatrix} V_1^-(p) & \frac{1}{3}[B(p) - \sqrt{2}\,G(p)]\sin\beta\,\boldsymbol{\sigma e}_p \\ \frac{1}{3}[B(p) + \sqrt{2}\,G(p)]\sin\beta\,\boldsymbol{\sigma e}_p & V_2^-(p) \end{vmatrix} \\ M_D = \dfrac{2}{\sqrt{\pi}} \begin{vmatrix} \dfrac{2}{\sqrt{5}}\sin^2\beta\, V_1^+(p) & \frac{1}{3}\left[B(p) + \dfrac{1}{\sqrt{2}}\,G(p)\right]\sin\beta\,\boldsymbol{\sigma e}_p \\ \dfrac{1}{3}\left[B(p) - \dfrac{1}{\sqrt{2}}\,G(p)\right]\sin\beta\,\boldsymbol{\sigma e}_p & \dfrac{2}{\sqrt{5}}\sin^2\beta\, V_2^+(p) \end{vmatrix} \end{cases}$$

It is then easy to calculate the proportion of S and D states by the integrals

$$p_S = \frac{\mathrm{Tr}\int |M_S|^2 p^3\,\mathrm{d}p\,\sin^2\beta\,\mathrm{d}\beta}{\mathrm{Tr}\int |\Psi(\mathbf{p})|^2\,\mathrm{d}\mathbf{p}},$$

$$p_D = \frac{\mathrm{Tr}\int |M_D|^2 p^3\,\mathrm{d}p\,\sin^2\beta\,\mathrm{d}\beta}{\mathrm{Tr}\int |\Psi(\mathbf{p})|^2\,\mathrm{d}\mathbf{p}}.$$

The integration over the angle β are immediate. After calculating the traces we obtain

$$p_S = \frac{N_S}{N_S + N_D} \qquad \text{and} \qquad p_D = \frac{N_D}{N_S + N_D},$$

with

$$N_S = \int_0^\infty [\,|V_1^-(p)|^2 + |V_2^-(p)|^2 + \tfrac{1}{6}|B(p)|^2 + \tfrac{1}{3}|G(p)|^2]p^3\,dp\,,$$

$$N_D = \int_0^\infty [\,|V_1^+(p)|^2 + |V_2^+(p)|^2 + \tfrac{1}{3}|B(p)|^2 + \tfrac{1}{6}|G(p)|^2]p^3\,dp\,.$$

5. – Numerical resolution of integral equations in the Ladder approximation.

As we saw before, the possibility to decouple the equations with respect to the index n focuses all the difficulties on the solving of the system of coupled integral equation (7). After fixing the binding energy of the deuteron at its experimental value, we intend to determine the coupling constant and the wave-functions $V_1^-(p)$ and $V_2^-(p)$ from which we can express all the others by quadratures.

5·1. *Approximate resolution.* – We substitute for the exact kernel $\Lambda_n(p, p')$ an approximate and separable one:

$$\Lambda_n(p, p') \to \frac{\Lambda_n(p, q)\,\Lambda_n(q, p')}{\Lambda_n(q, q)}\,,$$

choosing for q the average momentum of nucleons in the deuteron ($q \simeq \frac{1}{4}\mu$). We can then solve the new system exactly and we find $\alpha = 0.13$, *i.e.* $g^2/4\pi$, the coupling constant of nuclear forces, is equal to 16.

5·2. *Kellog's method of iteration.* – We have made a numerical calculation on IBM 650 to solve the equations by successive iterations. We thus define a series of values for α at each stage of the iteration. The result is

$$\alpha \simeq 0.046 \qquad i.e. \quad \frac{g^2}{4\pi} \simeq 6\,,$$

and the proportion of D state is

$$p_D \simeq 1\,\%\,.$$

We have verified that the development with respect to n is excellent as well. The solving of the system (13) instead of (7) leads us to

$$\varkappa \simeq 0.048 \qquad i.e. \quad \frac{g^2}{4\pi} \simeq 6.5 \, .$$

We then obtain a clear disagreement with the experimental value $g/4\pi \sim 15$ deduced for example from meson-nucleon scattering.

The most reasonable explanation for this seems to be the inability of the ladder approximation to describe this phenomenon. It would be necessary in a meson theory to include fourth order terms. Unfortunately this calculation is inextricable and we shall not do it although the formalism remains valid.

6. – Approximate wave functions.

The calculations with a kernel including the fourth order term being inextricable, we have modified our method in order to obtain a deuteron wave matrix available in concrete problems [5].

If we consider Bethe and Salpeter's equation:

$$\Phi(\mathbf{p}) = \frac{\varkappa}{(\gamma^{(1)} p_1 - iM)(\gamma^{(2)} p_2 - iM)} \gamma_5^{(1)} \gamma_5^{(2)} \int W(\mathbf{p}, \mathbf{p}') \, \Phi(\mathbf{p}') \, d\mathbf{p}' \, ,$$

we can see that there are 3 types of relativistic corrections:

— the kinematical corrections take the relativistic energy and the spins of the nucleons into account, they are included in the expressions $(\gamma^{(1)} p_1 - iM)$ and $(\gamma^{(2)} p_2 - iM)$;

— the properties of symmetry are correctly represented by the matrices $\gamma_5^{(i)}$ which show that we have used a pseudo-scalar meson-nucleon interaction;

— the dynamical corrections are included in the expression of the kernel $W(\mathbf{p}, \mathbf{p}')$.

In our previous calculation, we have precisely taken the first two kinds of correction into account. On the other hand, we have only partially considered the relativistic effects in the dynamical corrections, assuming the hypothesis that the interaction depends only upon the square of the space-time distance between the two particles. In particular, we have used the ladder approximation.

In separating the two types of corrections, we have obtained:

(5) We are indebted to Professor M. LÉVY for this suggestion.

a) a correct structure of the wave matrix—*i.e.* the distribution between its different components—due only to the spin of the nucleons and to the pseudo-scalar character of the interaction;

b) the wave functions $V_1^-(p)$, $V_2^-(p)$, which are approximate, but which take some relativistic effects into account.

In order to simplify the integral equations, we keep only the leading terms, and this enables us to obtain a simpler structure of the wave matrix. Considering that

$$\left(\frac{\gamma}{2M}\right)^2 \ll 1$$

and

$$V_1^-(p)V^-(p) = \quad \cdot B^2(p)$$

with a very good approximation, we can obtain the wave matrices M_S and M_D

$$(18)\quad \begin{cases} M_S = \sqrt{\frac{2}{\pi}} \begin{vmatrix} 1 & -\frac{1}{3}\left(\frac{p}{2M}\right)\left(1-\frac{h}{3}\right)\sin\beta(\boldsymbol{\sigma}\boldsymbol{e}_p) \\ \frac{p}{2M}\left(1-\frac{h}{3}\right)\sin\beta\boldsymbol{\sigma}\boldsymbol{e}_p & -\left(1-\frac{h}{3}\right)^2\left(\frac{p}{2M}\right)^2 \end{vmatrix} V_1^-(p) \\ M_D = \frac{2}{\sqrt{\pi}} \begin{vmatrix} -\frac{8}{3}\frac{1-(h/2)}{1+h}\left(1-\frac{h}{3}\right)^2\left(\frac{p}{2M}\right)^2\sin^2\beta & \frac{2}{3}\left(\frac{p}{M}\right)\left(1-\frac{h}{3}\right)\sin\beta\boldsymbol{\sigma}\boldsymbol{e}_p \\ 0 & \frac{2}{3}\left(1-\frac{h}{3}\right)\left(\frac{p}{2M}\right)^2\sin^2\beta \end{vmatrix} V_1^-(p) \end{cases}$$

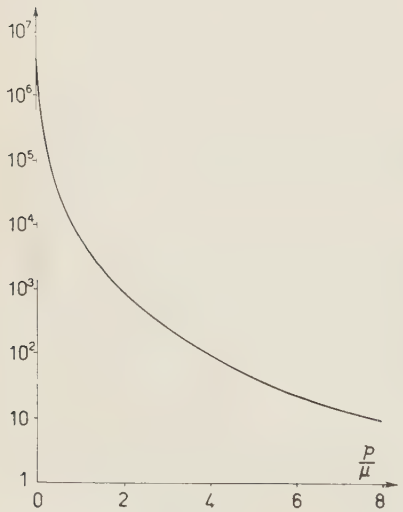

Fig. 1.

The only remaining unknown scalar function $V_1^-(p)$ corresponds, in the non-relativistic approximation, to the wave function for a deuteron without spin. We represent it graphically in Fig. 1.

The calculation of the immixture of D state gives 4%, which is in satisfactory agreement with experiments. We think that this very phenomenological determination of the deuteron wave function is better than the preceding one. It amounts to writing the wave matrix with the help of a scalar function resembling that of Hulthén, and known matrices describing a structure bound to the existence of nucleon spin. In other words, the matrices M_S and

M_D express the coupling between the states ${}^3S_1$ and ${}^3D_1$ of the deuteron and include relativistic effects inside the deuteron which are included in $V_1^-(p)$ a function of the same type as $A_0(p)$, the solution corresponding to the problem without spins.

7. – Conclusion.

We have developed a formalism to determine the relativistic deuteron wave function and the coupling constant of nuclear forces from the binding energy of the deuteron which has been experimentally determined.

We have used the interaction given by the meson theory about nuclear forces in the ladder approximation of second order. The theoretical results make us foresee a coupling constant twice too small and this does not allow us to take too seriously the results obtained about the wave function.

We have tried to consider the mass of the meson π as a phenomenological parameter. The most favourable results, corresponding to $\mu = 0.1$ ($g^2/4\pi \simeq 7$ and $p_D \simeq 1.5\%$) show that the coupling constant as well as the proportion of D state are parameters which vary extremely slowly with μ. Besides, this has been verified in the scattering problem.

We have also thought of adjoining an arbitrary phenomenological term respecting the conditions of invariance to the mesonic term of second order; an attempt has been made but was not very conclusive; perhaps, there are after all, some attempts to be made in this direction, but the justification and the theoretical origin of such terms seem to be extremely problematical and finally doomed to failure.

With extremely weak approximations, it is possible to give to the wave matrix a structure entirely known from one scalar function only. This method leads us to a very simple form and will permit numerical computations. Particularly, the most immediate and interesting application is the calculation of the elastic electron-deuteron scattering cross-section in the impulse approximation. The problem becomes very clear to understand. We know how the interaction acts on the neutron and on the proton and all the trace calculations can be performed before integration. The modifications due to the spin of the nucleons will appear naturally and the relativistic corrections will be rapidly evaluated.

RIASSUNTO (*)

Continuando la nostra ricerca delle funzioni d'onda relativistiche del deutone, abbiamo applicato l'equazione d'onda relativistica di Bethe e Salpeter al sistema legato di due particelle con spin $\frac{1}{2}$. Introdotte alcune lievissime approssimazioni, la soluzione del sistema infinito di equazioni integrali accoppiate ci consente di trovare la struttura esplicita delle matrici d'onda corrispondenti agli stati S e D. La percentuale di stati D ottenuta è in buon accordo con il risultato sperimentale.

(*) *Traduzione a cura della Redazione.*

Theory of Low Energy Nucleon-Nucleon Scattering.

III. Partial Wave Integral Equations for Low Orbital Momentum Amplitudes.

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Summary. — The theory developed in our previous papers ([1-2]) is applied to the low orbital momentum scattering amplitudes. Integral equations for these amplitudes are constructed, and the entire two-meson contribution to the left hand cut is explicitly given in terms of the effect of the Born term and 3, 3 resonance in pion-nucleon scattering and a possible π-π interaction. The contributions to the cut from higher mass intermediate states are assumed to be given by a low order polynomial in the energy, whose coefficients will appear as phenomenological constants in the theory. The integral equations are exactly of the form treated recently by Noyes ([3]) and the present work may be regarded as a specification of the left hand cut discontinuity to be used in those equations, whose solutions should give the energy variation of the low partial waves.

1. – Introduction.

In a previous paper ([1]) (henceforth referred to as I) we have developed a theory of low energy nucleon-nucleon scattering, based on a two-dimensional representation of the nucleon-nucleon scattering amplitudes. It was shown that a convenient set of scalar amplitudes $c_j^T(wt\bar{t})$ could be found which had simple crossing properties and which satisfied a Mandelstam representation, at least up to 4-th order perturbation theory. The spectral functions of this representation were calculated in terms of functions describing pion-nucleon and

(*) Elsie Ballot fellow on leave of absence from St. John's College, Cambridge.

([1]) D. Amati, E. Leader and B. Vitale: *Nuovo Cimento*, **17**, 68 (1960).

([2]) D. Amati, E. Leader and B. Vitale: *Nuovo Cimento*, **18**, 209 (1960).

pion-pion scattering. In particular the role of a possible π-π resonance in an s or p state was analysed.

In a second paper ([2]) (henceforth referred to as II) the simplification arising in the case of « high » angular momentum states was explicitly given.

In the present paper we show how in the case of « low » angular momentum states one can use the representation of the $c_j^T(wt\bar{t})$ in order to write down integral equations for the partial wave amplitudes of nucleon-nucleon scattering.

The type of integral equations which result, in which the jump on the left-hand cut is known (at least in terms of some parameters of the theory) and in which the jump of the right-hand cut is expressible in terms of a real phase, have been analysed recently by NOYES ([3]), who has discussed a method for their solution. As NOYES has remarked, the specification of a particular meson theory approximation or of a phenomenological model will just be equivalent to the specification of the jump on the left-hand cut in the integral equations for the partial wave amplitudes.

The aim of this paper is, therefore, to give explicitly the form of this jump in the two-meson approximation, exhibiting separately the contributions which arise from the fourth order perturbation terms, the pion-nucleon rescattering corrections, and the effect of a pion-pion interaction.

In Section 2 we discuss the analytic properties of the singlet and triplet partial wave amplitudes as derived from the properties of the $c_j^T(wt\bar{t})$ amplitudes, and present the results for the jump on the left-hand cut.

Section 3 considers the extension of these properties to those of the scattering amplitudes and discusses the validity of the approximations of the theory.

We shall throughout the paper refer to the explicit formulae contained in I and II.

2. – Analytic properties of the partial wave amplitude.

2·1. *Properties of the $c_j^T(wt\bar{t})$ functions.* – It can be easily seen from (I-3.4), (I-3.7) and (I-3.43) that the functions $c_j(wt\bar{t})$ are analytic functions in the cut w plane, with cuts only on the real axis. The discontinuity across the left cut can be expressed in terms of the functions ϱ defined in (I-3.43), in which all the dependence on the pion-nucleon and pion-pion interaction is made ex-

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([3]) H. R. NOYES: *Energy dependence of the nucleon-nucleon phase shifts*, UCRL-5921-T (Livermore). See also: G. F. CHEW and S. MANDELSTAM: UCRL-8728; H. P. NOYES and D. Y. WONG: *Phys. Rev. Lett.*, **3**, 191 (1959); *A new non-relativistic approximation for the two nucleon interaction due to single pion exchange*, UCRL (unpublished).

plicit, and the discontinuity across the right cut is connected with the imaginary part of the function c_j for positive values of k^2.

As was mentioned at the end of I the right hand discontinuities of the $c_j(wt\bar{t})$ are (apart from the 4-th order perturbation theory contributions) only weakly dependent on $t-\bar{t}$ (or equivalently on $\cos\theta$, where θ is the nucleon-nucleon C.M. scattering angle), so that in the projection into partial wave amplitudes it was possible, for the « high » waves, to retain only the 4-th order contribution to the right-hand discontinuity.

In the case of the « low » waves it is of course essential to retain the entire right-hand jump, and to determine it by using the unitarity condition in the nucleon-nucleon channel (*). This will relate the right-hand discontinuity to the imaginary part of the amplitude, which in turn can be related to the phase-shifts and coupling parameters. There will thus result, instead of the explicit formulae of the high wave case, a set of integral equations for the low wave amplitudes.

Concerning the left-hand cut in the k^2 plane, it should be realized that our knowledge of the jump is limited to that part of the negative real axis where the one and two pion contributions predominate. This region is, however, just that one in which the energy dependence is strong. What is left of the negative real axis will therefore contribute only a weak dependence on k^2. This means that we shall not be able to determine *completely* the low angular momentum amplitudes (as was indeed possible for the high waves), but that we can hope to be able to give their energy dependence, determining therefore from our theory their effective ranges and shape dependent parameters. The physical reason for this is rather clear: low waves are sensitive to the inner region of the interaction, which is just determined by the far lying singularities; because of our lack of knowledge on these singularities, our knowledge of the low wave amplitudes will always be at least within a constant.

2·2. *Properties of the singlet and triplet amplitudes.* – The scattering amplitudes α_i^T (as defined by Blatt and Biedenharn ([4]) or Stapp, Ypsilantis and Metropolis ([5])) are simply related to the partial wave singlet and triplet amplitudes T_{ij}^T (see Goldberger, Nambu and Oehme ([6])) which arise very naturally in the dispersion relation treatment of nucleon-nucleon scattering. They are in fact just linear combinations (with *numerical* coefficients) of $kT_{ij}^T(l)$.

(*) We stress once more that the division into « low » or « high » waves is somewhat arbitrary *a priori*, but is determined *a posteriori* by consistency requirements. We shall assume (as in II) that the « low » waves are those for which $l<2$.

([4]) J. M. Blatt and L. C. Biedenharn: *Rev. Mod. Phys.*, **24**, 258 (1952) and *Phys. Rev.*, **86**, 399 (1952).

([5]) H. P. Stapp, T. Ypsilantis and N. Metropolis: *Phys. Rev.*, **105**, 302 (1957).

([6]) M. L. Goldberger, Y. Nambu and R. Oehme: *Ann. of Phys.*, **2**, 226 (1957).

The $T^T_{ij}(l)$ are defined in terms of the total singlet-triplet amplitudes $T^T_{ij}(w \cos\theta)$ by

$$T^T_{ij}(l) = \frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\int_{-1}^{+1} T^T_{ij}(w, \cos\theta)\, P^m_l(\cos\theta)\, \mathrm{d}(\cos\theta)\,, \tag{2.1}$$

with $m = |i-j|$.

The $T^T_{ij}(w, \cos\theta)$ on the other hand are related to the $c^T_j(wt\bar{t})$ and therefore to the perturbative invariants $p^{\pm}_j(wt\bar{t})$ (see I, equations (2.32), (3.4), (3.5)) by

$$T^T_{ij}(w, \cos\theta) = \sum_\beta W^\beta_{ij} p^T_\beta(wt\bar{t})\,, \tag{2.2}$$

where

$$W^\beta_{ij}(w, \cos\theta) = \sum_\alpha V^\alpha_{ij} U_{\alpha\beta} \tag{2.3}$$

and the matrices V^α_{ij}, $U_{\alpha\beta}$ are given at the end of II, Section **3** (*).

Let us define

$$v = k^2 \qquad \text{and} \qquad \cos\theta = c\,,$$

so that

$$\begin{cases} w = 4m^2 + 4v\,, \\ t = -2v(1-c)\,, \\ \bar{t} = -2v(1+c)\,, \end{cases} \tag{2.4}$$

The matrix W^β_{ij}, since it contains terms in $\sqrt{w}$, will introduce an extra cut with branch point at $v = -m^2$ when going from the $p^{\pm}(wt\bar{t})$ to the $T_{ij}(w, c)$. However, in so far as our discussion is concerned, this is a very far-lying singularity and will not affect our results, so that for our purpose the $T_{ij}(w, c)$ may be considered to have the same analytic behaviour as the $p^{\pm}(wt\bar{t})$, apart, of course, from the deuteron pole whose contribution to the $T^T_{ij}(w, c)$ has been given in reference ([6]), and which occurs only in the triplet, $T = 0$ amplitudes.

We had shown in (I-3.7) that the $p^{\pm}(wt\bar{t})$ could be represented as

$$p^{\pm}_j(wt\bar{t}) = \frac{P^{\pm}_j}{\mu^{2-t}} + \frac{1}{\pi}\int_{4\mu^2}^{t'\,\max} \frac{\varrho^{\pm}_j(wt')}{t'-t}\,\mathrm{d}t' + \frac{1}{\pi}\int_{4m^2}^{\infty} \frac{\chi^{\pm}_j(x, t)}{x-w}\,\mathrm{d}x \mp (-1)^j (w \to \bar{t})\,. \tag{2.5}$$

(*) One must simply replace y by $\cos\theta$ in these formulae for $U_{\alpha\beta}$ and V^α_{ij}. Recall also that the p^π are given by $p^1 = 3p^+ + 2p^-$, $p^0 = 3p^+ - 6p^-$.

The limit $t'_{\max}$ as discussed in II should strictly be $= 9\mu^2$ for an exact two-pion treatment, but because of the slow threshold increase of the three pion contribution beginning at $9\mu^2$, it is reasonable to take $t'_{\max}$ greater than $9\mu^2$ since the two pion contribution will still be the dominant one.

The weight functions $\varrho_i^{\pm}(wt')$ of (2.5) have been given explicitly in II (see equations (3.13, 16, 17, 18) and (4.17)) where they were split up into various terms arising from the effect in pion-nucleon scattering of the Born term, 3,3 resonance, and a possible π-π interaction.

Using these formulae for the $\varrho_j^{\pm}(w, t')$, and equations (2.5), (2.2) and (2.1), we are now in a position to give in detail the analytic properties of the $T_{ij}^I(l)$. One sees that the $T_{ij}^I(l)$ are analytic functions of ν in the cut ν plane with the following singularities on the real axis:

a) The one meson exchange term (pole contribution in (2.5)) gives rise to a branch line along the negative axis, with branch point at $-\mu^2/4$ and the discontinuity $2i\,\mathrm{Im}\,T$ across this cut is given by

$$+\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\pi i\sum_{\beta}\frac{1}{\nu}P^{\pm}W_{ij}^{\beta}(\nu, c')P^m(c')\,, \tag{2.6}$$

with

$$c' = \frac{2\nu+\mu^2}{2\nu}$$

and the residues $P^{\pm}$ are given in (I-3.7).

b) The integral

$$\int\frac{\varrho}{t'-t}\,\mathrm{d}t'\,,$$

gives rise to a branch line along the negative axis, with branch point at $-\mu^2$. The two-pion contribution to the discontinuity $2i\,\mathrm{Im}\,T$ across this cut is given by

$$+\frac{2l+1}{2}\frac{(l-m)!}{(l+m)!}\,i\sum_{\beta}\frac{1}{\nu}\int_{4\mu^2}^{-4\nu}\varrho_{\beta}^{\pm}(wt')W_{ij}^{\beta}(\nu, y)\,P_l^m(y)\,\mathrm{d}t' = \tag{2.7}$$

$$= -(2l+1)\frac{(l-m)!}{(l+m)!}\,i\sum_{\beta}\int_{-1}^{1+\frac{2\mu^2}{\nu}}\varrho_{\beta}^{\pm}(wt)W_{ij}^{\beta}(\nu, c)\,P_l^m(c)\,\mathrm{d}c\,,$$

for

$$-\frac{t'_{\max}}{4}\leqslant\nu\leqslant-\mu^2\,,$$

where

$$y = \frac{t' + 2\nu}{2\nu} .$$

c) The crossed term $w \to \bar{t}$ in (2.5) gives rise to a branch line along the negative axis, but with branch point at $-m^2$. Such far lying singularities have been consistently disregarded in all our calculations as explained in I.

d) Both the second and the third term in (2.5) give rise to a branch line along the positive axis, with branch point at 0. The discontinuity across this cut will just be equal to $2i$ times the imaginary part of $T_{ij}(l)$ for positive values of ν.

If one now takes the correct combinations of $\pm$ amplitudes so as to form the amplitudes of definite T value (see note on page 461) then formulae (2.6), (2.7) together with the value of the $\varrho_{\beta}^{\pm}$ functions given in II, give explicitly the left-hand discontinuities of the $T_{ij}^{T}(l)$ amplitudes in the two-meson approximation.

Finally, it must be remembered that, in order to take account of the crossing $t \rightleftarrows \bar{t}$ of (I-3.4), one should multiply the discontinuities in the $T_{ij}^{T}(l)$ as given above by a factor

$$\nu(T, m, l, s) = 1 - (-1)^{l+m+T+s}, \tag{2.8}$$

where s is the total spin (*i.e.* 1 for triplet, 0 for singlet states) of the nucleon-nucleon system.

3. – Analytic properties of the scattering amplitudes.

From the above discussion of the $T_{ij}^{T}(l)$ and their relation with the scattering amplitudes $\alpha_l(\nu)$, it follows that the amplitudes.

$$a_l(\nu) = \sqrt{\frac{m^2 + \nu}{\nu}}\, \alpha_l(\nu) , \tag{3.1}$$

have the same singularities as the $T_{ij}(l)$.

Furthermore, since the $\alpha_l(\nu)$ go to zero as k^{2l+1} it follows that

$$a_l(\nu) \to 0 \qquad \text{as} \qquad \nu^l . \tag{3.2}$$

Thus we may divide the $a_l(\nu)$ by a factor ν^l without introducing any extra pole.

Finally, the set of amplitudes

$$b_l(\nu) = \frac{a_l(\nu)}{\nu^l} = \frac{1}{\nu^l}\sqrt{\frac{m^2+\nu}{\nu}}\,\alpha_l(\nu)\,, \tag{3.3}$$

will have no zeros (which will be useful for the application of the unitarity condition to the right cut, where the imaginary part of the inverse of the amplitude is required) and will have the required convergence properties (for $l \neq 0$). As a matter of fact, the unitarity condition implies that the α_l are bounded at infinity, so that the amplitudes b_l defined by (3.3) give rise to convergent integrals at infinity when $l \neq 0$. Only the s waves, therefore, will require convergence factors or, what is the same, they will require a subtraction.

It is now possible to write down for the amplitudes b_l a general integral representation of the form:

$$b_l(\nu) = \frac{D_l}{\nu_D - \nu} + \frac{1}{\pi}\int_{-\mu^2/4}^{\infty}\frac{\operatorname{Im} b_l(\nu')}{\nu' - \nu}\,\mathrm{d}\nu' + \frac{1}{\pi}\int_{0}^{\infty}\frac{\operatorname{Im} b(\nu')}{\nu' - \nu}\,\mathrm{d}\nu'\,, \tag{3.4}$$

where the discontinuity across the left cut is known, as already discussed, and that across the right cut is connected through unitarity to the phase shifts and coupling parameter in b_l.

Our determination of the discontinuity across the left cut of the partial amplitudes b_l is limited, as already emphasized, to the two-meson approximation, which means that our jumps are exact only down to $-(9/4)\mu^2$, where the three pion cut begins. We have already noted in Section 2·2 that the validity of this approximation can be reasonably extended beyond this value, down to a value $-t'_{\max}/4$ (see eq. (2.7)). The final results will be of course insensitive to the actual choice of $t'_{\max}$, as the contributions due to a variation in the value of $t'_{\max}$ will be just lumped into the low power polynomial in ν that will describe phenomenologically what remains of the left cut discontinuity—namely, that going from $-t'_{\max}/4$ to $-\infty$. The actual number of free constants to be so introduced in the theory will clearly depend only on the possibility of a good fit with experiment.

We note also that our assumption that the negative jump for large values of ν can be expressed by means of a low power polynomial is coincident with the hypothesis that the three pion contribution to the spectral function varies slowly with the energy. Such would not be the case, for instance, if a three pion resonance would turn out to be dominant in a region near the three pion threshold.

The discontinuity across the right cut can be expressed, through unitarity,

in terms of the phase shifts and coupling parameters that define the elastic nucleon-nucleon scattering. This is readily done for the uncoupled partial wave amplitudes (singlet and triplet with $j=l$), where α is of the form $\exp[i\delta]\sin\delta$ but represents a more complex problem in the case of coupled triplet amplitudes, with $j=l+1$ and $j=l-1$.

This procedure is valid only as long as the phase shifts are real, which means below the threshold for inelastic collisions. As the inelastic cross-section is still rather small at energies well above threshold, it could be safely assumed that the phase shifts are almost real also in a region of energy above the inelastic threshold. Thus what remains of the right cut will contribute only a weak dependence on ν in the partial wave amplitudes, and can be lumped into the constant or constants introduced already, in the phenomenological treatment of the left cut for large negative values of ν. The integral equations for the phase shifts and coupling parameters which result from equation (3.4) are exactly of the type discussed by NOYES [3] and the reader is referred to that paper for the method of solution.

A determination of the energy dependence of the singlet phase shifts is at present in progress, using the left-cut discontinuities given in the present paper.

RIASSUNTO

In lavori precedenti [1,2] è stata sviluppata una teoria della diffusione nucleone-nucleone a bassa energia; questa teoria è applicata nel presente lavoro alla analisi delle ampiezze di transizione corrispondenti a bassi valori del momento orbitale. Si costruiscono equazioni integrali per queste ampiezze e si calcola l'intero contributo degli stati intermedi con due mesoni alla discontinuità delle ampiezze per valori negativi di k^2. Questo contributo è dato in modo esplicito in funzione del termine calcolato nell'approssimazione di Born con costante di accoppiamento rinormalizzata, della risonanza 3.3 nella diffusione pione-nucleone e di una possibile risonanza nella diffusione pione-pione in stato s o p. I contributi a questa discontinuità a masse intermedie più alte sono approssimati mediante un polinomio in cui sono presenti solo basse potenze di k^2 e i cui coefficienti appaiono come costanti fenomenologiche nella nostra teoria, costanti da determinare mediante il confronto con l'esperienza. Le equazioni integrali così ottenute sono del tipo recentemente discusso in dettaglio da NOYES [3]. Il presente lavoro può essere considerato come una determinazione della discontinuità delle ampiezze di diffusione per valori negativi di k^2, discontinuità che verrà poi usata per risolvere le equazioni integrali. A causa delle costanti fenomenologiche introdotte per descrivere le singolarità lontane, la presente teoria non può dare il valore delle fasi per le onde di basso momento orbitale, ma può invece portare ad una determinazione della loro dipendenza dalla energia, e permettere quindi di determinare il « raggio effettivo » e il « parametro dipendente dalla forma », che, insieme con la « lunghezza di diffusione », permettono di descrivere la diffusione nucleone-nucleone a bassa energia.

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Etude d'un modèle de champ à constante de renormalisation nulle.

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Résumé. — Il est montré sur un exemple soluble qu'une théorie à couplage de Fermi est équivalente à une théorie à couplage de Yukawa dans laquelle la constante de renormalisation de la particule intermédiaire est nulle. L'annulation de cette constante est ensuite interprétée dans un formalisme lagrangien comme une contrainte imposée aux champs. On étudie enfin la singularité des relations de commutation.

1. – Introduction.

G. Källén [1] a donné des arguments tendant à prouver que certaines constantes de renormalisation de l'électrodynamique sont nulles. D'autre part A. S. Wightman [2] discute brièvement mais d'un point de vue plus général le même cas de nullité.

Il nous a paru intéressant d'examiner ce qui se produit dans une théorie où ce cas singulier se présente et spécialement d'étudier, comme A. S. Wightman, ce que deviennent les relations de commutation des champs renormalisés qui, à priori, perdent leur sens.

Notre point de départ se trouve dans un théorème d'équivalence établi par B. Jouvet [3], selon lequel toute théorie à couplage de Fermi est équivalente à une théorie à couplage de Yukawa pour laquelle la constante de renormalisation de la particule intermédiaire est nulle.

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(1) G. Källén: *Dan. Mat. Fys. Medd.*, **27**, no. 12 (1953).
(2) A. S. Wightman: *Phys. Rev.*, **101**, 860 (1956).
(3) B. Jouvet: *Nuovo Cimento*, **5**, 1 (1957).

Afin que tout soit entièrement calculable nous utiliserons des modèles simples inspirés par le modèle de Lee et nous commencerons par donner une démonstration directe du théorème d'équivalence mettant en évidence la condition $Z = 0$ (Sections **2** et **3**).

Cette condition sera introduite ensuite (Section **4**) dans un formalisme lagrangien qui montrera que l'annulation de la constante de renormalisation signifie une interdépendance des champs intervenant dans la théorie à couplage de Yukawa.

Dans un dernier paragraphe (Section **5**) on étudie les propriétés de la function spectrale de H. LEHMANN dans le cas $Z = 0$.

2. – Modèle à couplage de Yukawa (modèle de Lee).

Nous considérons le modèle de théorie à trois champs ψ_V, ψ_N et A défini par l'hamiltonien renormalisé de Lee [4,5]

$$H_1 = Zm_V \int d\boldsymbol{p}\, \psi_V^+(\boldsymbol{p})\, \psi_V(\boldsymbol{p}) + m_N \int d\boldsymbol{q}\, \psi_N^+(\boldsymbol{q})\, \psi_N(\boldsymbol{q}) + \int d\boldsymbol{k}\, \omega_k a^+(\boldsymbol{k})\, a(\boldsymbol{k}) -$$

$$- \frac{g}{(2\pi)^{\frac{3}{2}}} \int d\boldsymbol{p}\, d\boldsymbol{q}\, d\boldsymbol{k}\, \frac{f(\omega_k)}{\sqrt{2\omega_k}}\, \delta(\boldsymbol{p} - \boldsymbol{q} - \boldsymbol{k}) \left[\psi_V^+(\boldsymbol{p})\, \psi_N(\boldsymbol{q})\, a(\boldsymbol{k}) + a^+(\boldsymbol{k})\, \psi_N^+(\boldsymbol{q})\, \psi_V(\boldsymbol{k})\right] -$$

$$- \delta v \int d\boldsymbol{p}\, \psi_V^+(\boldsymbol{p})\, \psi_V(\boldsymbol{p})\,, \qquad (\omega_k = \sqrt{\boldsymbol{k}^2 + \mu^2})\,.$$

Les champs renormalisés ψ_V, ψ_N et A correspondent respectivement aux deux fermions V et N et au boson θ et satisfont aux relations de commutation et d'anticommutation usuelles

$$\{\psi_V(\boldsymbol{p}),\, \psi_V^+(\boldsymbol{p}')\} = \frac{1}{Z}\, \delta(\boldsymbol{p} - \boldsymbol{p}')\,, \qquad \{\psi_N(\boldsymbol{q}),\, \psi_N^+(\boldsymbol{q}')\} = \delta(\boldsymbol{q} - \boldsymbol{q}')\,,$$

$$[a(\boldsymbol{k}),\, a^+(\boldsymbol{k}')] = \delta(\boldsymbol{k} - \boldsymbol{k}')\,,$$

les autres commutateurs ou anticommutateurs étant nuls.

Ce modèle dépend des quatre constantes m_V, m_N, μ et g. Nous nous placerons dans le cas où la particule V est stable, c'est-à-dire où la relation $m_V < m_N + \mu$ est vérifiée.

Les constantes de renormalisation Z et δv ont alors pour expression

$$Z = 1 - \frac{g^2}{4\pi^2} \int_0^\infty dk\, \frac{k^2 f^2(\omega_k)}{\omega_k [\omega_k + m_N - m_V]^2}\,, \qquad \delta v = - \frac{g^2}{4\pi^2} \int_0^\infty dk\, \frac{k^2 f^2(\omega_k)}{\omega_k [\omega_k + m_N - m_V]}\,.$$

(4) T. D. LEE: *Phys. Rev.*, **95**, 1329 (1954).

(5) G. KÄLLÉN and W. PAULI: *Dan. Mat. Fys. Medd.*, **30**, no. 7 (1955).

Elles sont finies si la fonction de coupure f décroit suffisamment vite à l'infini ce que l'on supposera réalisé dans la suite. Nous supposerons aussi que Z n'est pas négatif de façon à exclure le cas où l'on aurait un fantôme.

Le modèle précédent correspond aux deux processus d'interaction du type de Yukawa $\mathrm{V} \rightleftarrows \mathrm{N} + \theta$ et possède de ce fait les deux constantes du mouvement

$$N_1 = N_{\mathrm{V}} + N_{\mathrm{N}}\,, \qquad N_2 = N_{\mathrm{N}} + N_{\theta}\,,$$

N_{V}, N_{N} et N_{θ} représentant respectivement les nombres de particules nues V, N et θ.

Dans le secteur $N_1 = N_2 = 1$ engendré par les vecteurs $\sqrt{Z}\psi_{\mathrm{V}}^{+}(\boldsymbol{p})\,|0\rangle$ et $\psi_{\mathrm{N}}^{+}(\boldsymbol{q})\,a^{+}(\boldsymbol{k})\,|0\rangle$ l'hamiltonien H_1 possède un système complet d'états propres qui sont:

$$|v_{\boldsymbol{p}}\rangle = Z\psi_{\mathrm{V}}^{+}(\boldsymbol{p})\,|0\rangle + \frac{g}{4\pi^{\frac{3}{2}}}\int \mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{k}\,\delta(\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{k})\,\frac{f(\omega_k)}{\sqrt{\omega_k}[\omega_k + m_{\mathrm{N}} - m_{\mathrm{V}}]}\cdot\psi_{\mathrm{N}}^{+}(\boldsymbol{q})a^{+}(\boldsymbol{k})\,|0\rangle,$$

$$|N_{\boldsymbol{q}_1}\theta_{\boldsymbol{k}_1}\rangle^{(\pm)} = \psi_{\mathrm{N}}^{+}(\boldsymbol{q}_1)a^{+}(\boldsymbol{k}_1)\,|0\rangle - \frac{gZ}{4\pi^{\frac{3}{2}}}\,J^{(\pm)}(k_1)\int \mathrm{d}\boldsymbol{p}\,\delta(\boldsymbol{p}-\boldsymbol{q}_1-\boldsymbol{k}_1)\,\psi_{\mathrm{V}}^{+}(\boldsymbol{p})\,|0\rangle -$$

$$-\frac{g^2}{2(2\pi)^3}\,J^{(\pm)}(k_1)\int \mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{k}\,\delta(\boldsymbol{q}+\boldsymbol{k}-\boldsymbol{q}_1-\boldsymbol{k}_1)\,\frac{f(\omega_k)}{\sqrt{\omega_k}[\omega_k - \omega_{k_1} \mp i\varepsilon]}\,\psi_{\mathrm{N}}^{\mp}(\boldsymbol{q})\;a^{\mp}(\boldsymbol{k})\,|0\rangle\,,$$

$|0\rangle$ représente le vide des champs libres. Il est identique au vide physique défini par la condition $H_1|0\rangle = 0$.

La fonction $J^{(\pm)}(k_1)$ est donnée par

$$J^{(\pm)}(k_1) = \frac{f(\omega_{k_1})}{\sqrt{\omega_{k_1}}}\,\frac{1}{[\omega_{k_1} + m_{\mathrm{N}} - m_{\mathrm{V}}]\,[1 + (g^2/4\pi^2)\;(\omega_{k_1} + m_{\mathrm{N}} - m_{\mathrm{V}})\sigma^{(\pm)}(k_1)]}\,,$$

avec

$$\sigma^{(\pm)}(k_1) = \int_0^{\infty} \mathrm{d}k\,\frac{k^2 f^2(\omega_k)}{\omega_k[\omega_k \pm m_{\mathrm{N}} - m_{\mathrm{V}}]^2[\omega_k - \omega_{k_1} \mp i\varepsilon]}\,.$$

Le premier état est un état lié correspondant à la particule V physique. Les autres états sont des états de diffusion des deux particules N et θ correspondant à des ondes diffusées purement sortantes ou entrantes.

On en déduit la matrice S décrivant la diffusion de N par θ, seul processus possible dans ce secteur:

$$\langle \boldsymbol{q}_2, \boldsymbol{k}_2|S_1|\boldsymbol{q}_1, \boldsymbol{k}_1\rangle = {}^{(-)}\langle N_{\boldsymbol{q}_2}\theta_{\boldsymbol{k}_2}|N_{\boldsymbol{q}_1}\theta_{\boldsymbol{k}_1}\rangle^{(+)} =$$

$$= \delta(\boldsymbol{q}_1 - \boldsymbol{q}_2)\,\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2) - \frac{ig^2}{8\pi^2}\,\frac{f(\omega_{k_1})}{\sqrt{\omega_{k_1}}}\,J^{(+)}(k_1)\,\delta(\boldsymbol{q}_1 + \boldsymbol{k}_1 - \boldsymbol{q}_2 - \boldsymbol{k}_2)\,\delta[\omega_{k_1} - \omega_{k_2}]\,.$$

3. – Modèle à couplage de Fermi.

Au modèle précédent nous ferons correspondre le modèle à deux champs seulement ψ_N et A défini par l'hamiltonien

$$H_2 = m_N \int d\boldsymbol{q}\, \psi_N^+(\boldsymbol{q})\psi_N(\boldsymbol{q}) + \int d\boldsymbol{k}\, \omega_k a^+(\boldsymbol{k})a(\boldsymbol{k}) +$$

$$+ \frac{\lambda}{2(2\pi)^3} \int d\boldsymbol{q}_1\, d\boldsymbol{q}_2\, d\boldsymbol{k}_1\, d\boldsymbol{k}_2 \frac{F(\omega_{k_1})F(\omega_{k_1})}{\sqrt{\omega_{k_1}\omega_{k_1}}} \delta(\boldsymbol{q}_1 + \boldsymbol{k} - \boldsymbol{q}_2 - \boldsymbol{k}_2)\psi_N^+(\boldsymbol{q}_1)\psi_N(\boldsymbol{q}_2)a^+(\boldsymbol{k}_1)a(\boldsymbol{k}_2),$$

et les relations de commutation et d'anticommutation

$$\{\psi_N(\boldsymbol{q}), \psi_N(\boldsymbol{q}')\} = \delta(\boldsymbol{q} - \boldsymbol{q}'), \qquad [a(\boldsymbol{k}), a^+(\boldsymbol{k})] = \delta(\boldsymbol{k} - \boldsymbol{k}').$$

Ce modèle dépend des trois constantes m_N, μ et λ et représente l'unique processus du type de Fermi $N + \theta \to N + \theta$ d'où résulte l'existence des deux constantes du mouvement N_N et N_θ.

Dans le secteur $N_N = N_\theta = 1$ l'hamiltonien H_2 possède un système complet d'états propres de la forme

$$|Z\rangle = \int d\boldsymbol{q}\, d\boldsymbol{k}\, \delta(\boldsymbol{p} - \boldsymbol{q} - \boldsymbol{k})\varphi(\boldsymbol{k})\psi_N^+(\boldsymbol{q})a^+(\boldsymbol{k})|0\rangle.$$

La recherche de ces états conduit à écrire

$$H_2|Z\rangle = E|Z\rangle$$

et fournit pour $\varphi(\boldsymbol{k})$ l'équation

$$[E - m_N - \omega_k]\varphi(\boldsymbol{k}) = \frac{\lambda}{2(2\pi)^3} \cdot I \cdot \frac{F(\omega_k)}{\sqrt{\omega_k}},$$

avec

$$I = \int d\boldsymbol{k} \frac{F(\omega_k)}{\sqrt{\omega_k}} \varphi(\boldsymbol{k}).$$

Pour $E > m_N + \mu$ l'équation précédente admet toujours des solutions que l'on peut exprimer à l'aide d'ondes sortantes $(+)$ ou entrantes $(-)$ et qui sont

$$\varphi^{(\pm)}(\boldsymbol{k}) = \delta(\boldsymbol{k} - \boldsymbol{k}_1) - \frac{\lambda}{2(2\pi)^3\sqrt{\omega_k}} I^{(\pm)}(k_1) \frac{F(\omega_k)}{\omega_k - \omega_k \mp i\varepsilon},$$

avec

$$I^{(\pm)}(k_1) = \frac{F(\omega_{k_1})}{\sqrt{\omega_{k_1}}} \frac{1}{1 + \dfrac{\lambda}{4\pi^2}\displaystyle\int_0^\infty \mathrm{d}k \, \dfrac{k^2 F^2(\omega_k)}{\omega_k[\omega_k - \omega_{k_1} \mp i\varepsilon]}} .$$

Ceci fournit les états de diffusion des deux particules N et θ

$$|N_{q_1}\theta_{k_1}\rangle^{(\pm)} = \psi_N^{\pm}(\boldsymbol{q}_1)a^+(\boldsymbol{k}_1)|0\rangle -$$
$$- \frac{\lambda}{2(2\pi)^3} I^{(\pm)}(k_1)\int \mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{k}\,\delta(\boldsymbol{q}_1 + \boldsymbol{k}_1 - \boldsymbol{q} - \boldsymbol{k}) \frac{F(\omega_k)}{\sqrt{\omega_k}[\omega_k - \omega_{k_1} \mp i\varepsilon]} \psi_N^{\pm}(\boldsymbol{q})\, a^+(\boldsymbol{k})|0\rangle ,$$

l'énergie correspondante étant $E = m_N + \omega_{k_1}$.

Pour $E < m_N + \mu$ il n'y a d'état propre que si λ vérifie la double inégalité

$$\text{(A)} \qquad -\frac{1}{4\pi^2}\int_0^\infty \mathrm{d}k \, \frac{k^2 F^2(\omega_k)}{\omega_k[\omega_k - \mu]} < \frac{1}{\lambda} < 0 .$$

On a alors, à une constante près

$$\varphi(\omega_k) = -\frac{\lambda}{2(2\pi)^3\sqrt{\omega_k}} \frac{F(\omega_k)}{\omega_k + m_N - E_0},$$

la valeur propre E_0 étant l'unique solution de l'équation

$$\text{(B)} \qquad \frac{1}{\lambda} = -\frac{1}{4\pi^2}\int_0^\infty \mathrm{d}k \, \frac{k^2 F^2(\omega_k)}{\omega_k[\omega_k + m_N - E_0]} .$$

Ceci donne l'état propre normé

$$|W_p\rangle = C\int \mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{k}\,\delta(\boldsymbol{p} - \boldsymbol{q} - \boldsymbol{k}) \frac{F(\omega_k)}{\sqrt{\omega_k}[\omega_k + m_N - E_0]} \psi_N^{\pm}(\boldsymbol{q})\, a^+(\boldsymbol{k})|0\rangle ,$$

avec

$$C = \left[4\pi\int_0^\infty \mathrm{d}k \, \frac{k^2 F^2(\omega_k)}{\omega_k[\omega_k + m_N - E_0]^2}\right]^{-\frac{1}{2}} .$$

L'inégalité $E_0 < m_N + \mu$ indique que l'on est en présence d'un état lié.

Nous allons montrer maintenant que, sous la condition (A), il est possible de déterminer les constantes d'un modèle de Lee de façon à le rendre identique au modèle précédent. Le terme « identique » signifiera pour nous que les deux modèles présentent les mêmes phénomènes observables (l'étude étant ici réduite aux deux secteurs respectifs $N_N = N_\theta = 1$ et $N_1 = N_2 = 1$) c'est-à-dire que les deux conditions suivantes sont remplies:

a) Egalité des matrices S donnant la diffusion de N par θ.

b) Existence dans le second modèle d'un état lié susceptible d'être interprété comme représentant a particule V physique du modèle de Lee c'est-à-dire d'énergie $E_0 = m_V$.

D'une façon générale, s'il est admis que les états liés de l'hamiltonien correspondent aux pôles de la matrice S, la condition *b*) n'est qu'une simple conséquence de la condition *a*) qui est alors la seule nécessaire. C'est ce qui sera vérifié dans la suite.

Le problème actuel est donc de déterminer les deux constantes m_V et g en fonction de λ, m_N et μ de façon à réaliser l'égalité des matrices S.

La matrice S du second modèle se calcule aisément. Elle a pour expression

$$\langle \boldsymbol{q}_2, \boldsymbol{k}_2 | S_2 | \boldsymbol{q}_1, \boldsymbol{k}_1 \rangle = {}^{(-)}\langle N_{\boldsymbol{q}_2} \theta_{\boldsymbol{k}_2} | N_{\boldsymbol{q}_1} \theta_{\boldsymbol{k}_2} \rangle^{(+)} =$$

$$= \delta(\boldsymbol{q}_1 - \boldsymbol{q}_2)\,\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2) - \frac{i\lambda}{8\pi^2} \frac{F(\omega_{k_1})}{\sqrt{\omega_{k_1}}} I^{(+)}(k_1)\,\delta(\boldsymbol{q}_1 + \boldsymbol{k}_1 - \boldsymbol{q}_2 - \boldsymbol{k}_2)\,\delta[\omega_{k_1} - \omega_{k_2}]\,,$$

la comparaison avec S_1 donne immédiatement la condition d'égalité

$$g^2 f(\omega_{k_1}) J^{(+)}(k_1) \equiv \lambda F(\omega_{k_1}) I^{(+)}(k_1)\,.$$

Nous allons d'abord transformer l'espression de $I^{(\pm)}(k_1)$, en y introduisant la masse m_V cherchée, par l'identité

$$\frac{1}{\omega_k - \omega_{k_1} \mp i\varepsilon} \equiv \frac{1}{\omega_k + m_N - m_V} +$$

$$+ \frac{\omega_{k_1} + m_N - m_V}{(\omega_k + m_N - m_V)^2} + \frac{(\omega_{k_1} + m_N - m_V)^2}{(\omega_k + m_N - m_V)^2 [\omega_k - \omega_{k_1} \mp i\varepsilon]}\,.$$

On obtient alors

$$I^{(\pm)}(k_1) = \frac{F(\omega_{k_1})}{\sqrt{\omega_{k_1}}} \frac{1}{1 - \lambda A + (\omega_{k_1} + m_N - m_V)[\lambda B + (\lambda/4\pi^2)(\omega_{k_1} + m_N - m_V)\Sigma^{(\pm)}(k_1)]}\,,$$

où l'on a posé

$$\begin{cases} A = -\dfrac{1}{4\pi^2} \displaystyle\int_0^\infty \mathrm{d}k \, \frac{k^2 F^2(\omega_k)}{\omega_k [\omega_k + m_N - m_V]}\,, \\ B = \dfrac{1}{4\pi^2} \displaystyle\int_0^\infty \mathrm{d}k \, \frac{k^2 F^2(\omega_k)}{\omega_k [\omega_k + m_N - m_V]^2}\,, \\ \Sigma^{(\pm)}(k_1) = \displaystyle\int_0^\infty \mathrm{d}k \, \frac{k^2 F^2(\omega_k)}{\omega_k [\omega_k + m_N - m_V]^2 [\omega_k - \omega_{k_1} \mp i\varepsilon]}\,. \end{cases}$$

La condition d'égalité écrite plus haut impose tout d'abord que les deux fonctions de coupure soient identiques

$$f(\omega_k) \equiv F(\omega_k)\,, \tag{I}$$

ce qui entraine les relations

$$A = \frac{\delta v}{g^2}\,, \qquad B = \frac{1-Z}{g^2}\,, \qquad \Sigma^{(\pm)}(k) \equiv \sigma^{(\pm)}(k)\,.$$

Une seconde condition à remplir, exprimant que la fonction $I^{(\pm)}(k_1)$ possède, comme $J^{(\pm)}(k_1)$, les pôles correspondant au facteur $(\omega_{k_1} + m_N - m_V)$ est

$$1 - \lambda A = 0\,,$$

ce qui peut encore s'écrire

$$\frac{1}{\lambda} = \frac{\delta v}{g^2} = -\frac{1}{4\pi^2}\int_0^\infty \mathrm{d}k\, \frac{k^2 f^2(\omega_k)}{\omega_k[\omega_k + m_N - m_V]}\,. \tag{II}$$

La fonction $I^{(\pm)}(k_1)$ s'écrit alors

$$I^{(\pm)}(k_1) = \frac{1}{\lambda B}\cdot\frac{f(\omega_{k_1})}{\sqrt{\omega_{k_1}}}\cdot\frac{1}{(\omega_{k_1} + m_N - m_V)[1 + (1/4\pi^2 B)(\omega_{k_1} + m_N - m_V)\sigma^{(\pm)}(k_1)]}\,,$$

et la dernière condition d'égalité des deux matrices S est

$$g^2 = \frac{1}{B}\,,$$

c'est-à-dire

$$Z = 1 - \frac{g^2}{4\pi^2}\int_0^\infty \mathrm{d}k\, \frac{k^2 f^2(\omega_k)}{\omega_k[\omega_k + m_N - m_V]^2} = 0\,. \tag{III}$$

La relation (II) détermine univoquement la masse m_V en fonction des constantes λ, m_N et μ. On a vu en effet que la solution de cette équation existe et est unique si λ vérifie les inégalités (A). Il en résulte bien d'après (B) l'égalité $m_V = E_0$: les deux hamiltoniens possèdent alors deux états liés de même valeur propre. La relation (III) fournit enfin g^2 en fonction de λ, m_N et μ. En fait, cette relation ne contient λ que par l'intermédiaire de la masse $m_V(\lambda)$ déterminée par (II) et prend ainsi la forme d'une condition à satisfaire entre les constantes définissant le modèle de Lee cherché. Elle caractérise donc parmi

tous les modèles de Lee possibles ceux qui seront identifiables à des modèles du second type.

En résumé, si l'on considère les modèles des deux types suivants:

1) modèle d'hamiltonien H_1 vérifiant la relation $Z = 0$,

2) modèle d'hamiltonien H_2 avec $F = f$ et vérifiant les relations (A),

il y a identité de tout modèle du type 1) avec un modèle du type 2), et réciproquement de toute modèle du type 2) avec un modèle du type 1), les modèles identiques étant reliés entre eux par la relation (II).

Remarques.

1) La condition *b*) citée plus haut n'est pas suffisante pour entraîner l'identité des deux modèles. On peut voir en effet que si cette condition est seule exigée, les fonctions de coupure étant prises identiques, la matrice S du second modèle prend la forme de celle obtenue dans le modèle de Lee mais où la constante g^2 est remplacée par $2(2\pi)^3\ C^2$, C étant la constante de normalisation de l'état lié $|W_p\rangle$ du second modèle.

Pour être équivalente à la condition *a*), il manque donc à *b*), qui est une condition sur les masses, une condition sur les charges permettant de poser la relation $g^2 = 2(2\pi)^3\ C^2$ (qui est équivalente à $Z = 0$).

2) Remarquons que, en tenant compte des conditions (I) et (II), si l'on porte la valeur $Z = 0$ dans les expressions des états propres de l'hamiltonien H_1 on retrouve les expressions des états propres de l'hamiltonien H_2. Au contraire de ce qui se passe pour la relation d'anticommutation du champ ψ_V (voir paragraphe suivant) cette identification des états peut s'obtenir par le simple passage à la limite $Z \to 0$.

Un tel passage à la limite couche les états propres de l'hamiltonien de Lee sur le sous-espace engendré par les vecteurs $\psi_N^+(\boldsymbol{q})\, a^+(\boldsymbol{k})\,|0\rangle$ en éliminant les états contenant la particule V. Le modèle de Lee tend ainsi vers un modèle à deux particules seulement.

Nous allons retrouver cette propriété au prochain paragraphe mais sans avoir la possibilité de faire intervenir un passage à la limite.

4. – Examen de la condition $Z = 0$.

Nous venons de trouver la condition pour qu'un modèle de Lee soit représentable par un modèle à deux champs du type envisagé au dernier paragraphe. Cette condition, qui exprime que la constante de renormalisation

de la particule V est nulle, ne peut être acceptée telle quelle dans la théorie développée au paragraphe 2 car elle rend singulière la relation d'anticommutation du champ ψ_V.

Nous allons interpréter maintenant ce résultat dans un formalisme canonique du modèle de Lee.

Le modèle de Lee renormalisé peut se déduire du Lagrangien suivant, écrit dans l'espace des impulsions:

$$L = Zi\int \mathrm{d}\boldsymbol{p}\,\psi_V^+(\boldsymbol{p})\,\dot{\psi}_V(\boldsymbol{p}) - Zm_V\int \mathrm{d}\boldsymbol{p}\,\psi_V^+(\boldsymbol{p})\psi_V(\boldsymbol{p}) + i\int \mathrm{d}\boldsymbol{q}\,\psi_N^+(\boldsymbol{q})\,\dot{\psi}_N(\boldsymbol{q}) -$$

$$- m_N\int \mathrm{d}\boldsymbol{q}\,\psi_N^+(\boldsymbol{q})\psi_N(\boldsymbol{q}) + i\int \mathrm{d}\boldsymbol{k}\,a^+(\boldsymbol{k})\,\dot{a}(\boldsymbol{k}) - \int \mathrm{d}\boldsymbol{k}\,\omega_k a^+(\boldsymbol{k})a(\boldsymbol{k}) +$$

$$+ \frac{g}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{k}\,\delta(\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{k})\,\frac{f(\omega_k)}{\sqrt{2\omega_k}}\,[\psi_V^+(\boldsymbol{p})\psi_N(\boldsymbol{q})\,a(\boldsymbol{k}) + a^+(\boldsymbol{k})\,\psi_N^+(\boldsymbol{q})\psi_V(\boldsymbol{p})] +$$

$$+ \delta\nu\int \mathrm{d}\boldsymbol{p}\,\psi_V^+(\boldsymbol{p})\psi_V(\boldsymbol{p})\,,$$

(les champs dépendent aussi du temps mais cette dépendance n'est pas indiquée explicitement).

Les champs conjugués des champs $\psi_V(\boldsymbol{p})$, $\psi_N(\boldsymbol{q})$, $a(\boldsymbol{k})$ sont respectivement

$$\tilde{\omega}_V(\boldsymbol{p}) = Zi\psi_V^+(\boldsymbol{p})\,, \qquad \tilde{\omega}_N(\boldsymbol{q}) = i\psi_N^+(\boldsymbol{q})\,, \qquad \tilde{\omega}_a(\boldsymbol{k}) = ia^+(\boldsymbol{k})\,.$$

Il en résulte un hamiltonien qui n'est autre que H_1 et des relations canoniques de commutation et d'anticommutation qui sont celles de la Section 2.

Si $Z = 0$ on obtient alors $\tilde{\omega}_V(\boldsymbol{p}) = 0$ et la relation canonique d'anticommutation du champ ψ_V perd son sens. Ceci signifie en fait que le champ ψ_V n'est plus indépendant des champs ψ_N et A.

En effet, l'équation de Lagrange relative à ψ_V déduite du Lagrangien ci-dessus est

$$Z\left[i\frac{\mathrm{d}}{\mathrm{d}t}\psi_V(\boldsymbol{p}) - m_V\psi_V(\boldsymbol{p})\right] =$$

$$= -\delta\nu\psi_V(\boldsymbol{p}) - \frac{g}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{k}\,\delta(\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{k})\,\frac{f(\omega_k)}{\sqrt{2\omega_k}}\,\psi_N(\boldsymbol{q})\,a(\boldsymbol{k})\,,$$

ce qui donne pour $Z = 0$

$$\text{(IV)} \qquad \psi_V(\boldsymbol{p}) = -\frac{g}{(2\pi)^{\frac{3}{2}}\delta\nu}\int \mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{k}\,\delta(\boldsymbol{p}-\boldsymbol{q}-\boldsymbol{k})\,\frac{f(\omega_k)}{\sqrt{2\omega_k}}\,\psi_N(\boldsymbol{q})\,a(\boldsymbol{k})\,.$$

Le champ ψ_V n'a donc pas a être quantifié séparément et seules subsistent les relations de commutation et d'anticommutation des champs ψ_N et A. Il est, en outre, possible d'éliminer ψ_V de l'hamiltonien H_1 de manière à n'exprimer celui-ci qu'en fonction de ces deux champs. On vérifie aisément que la substitution donne alors un hamiltonien du type H_2 avec

$$F(\omega_k) \equiv f(\omega_k) \qquad \text{et} \qquad \lambda = \frac{g^2}{\delta\nu},$$

et nous retrouvons ainsi les conditions d'identification (I) et (II) .

Ainsi le modèle de Lee apparait-il comme identique au modèle précédemment introduit, s'il est soumis à la condition $Z = 0$ qui représente alors une contrainte entre les champs.

Il n'y a donc maintenant plus de raison apparente pour que la relation d'anticommutation de ψ_V soit encore singulière.

Passons en effet aux champs dans l'espace des positions. Ils sont définis par

$$\psi_V(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}\boldsymbol{p}\,\psi_V(\boldsymbol{p})\exp[i\boldsymbol{p}\cdot\boldsymbol{x}], \qquad \psi_N(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}\boldsymbol{q}\,\psi_N(\boldsymbol{q})\exp[i\boldsymbol{q}\cdot\boldsymbol{x}],$$

$$A(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \frac{\mathrm{d}\boldsymbol{k}}{\sqrt{2\omega_k}}\,a(\boldsymbol{k})\exp[i\boldsymbol{k}\cdot\boldsymbol{x}].$$

Nous poserons aussi

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^3}\int \mathrm{d}\boldsymbol{k} f(\omega_k)\exp[i\boldsymbol{k}\cdot\boldsymbol{x}],$$

et l'on obtient aisément, à partir de (IV), (en faisant réapparaître la variable t) la relation

$$\text{(V)} \qquad \psi_V(\boldsymbol{x}, t) = -\frac{g}{\delta\nu}\,\psi_N(\boldsymbol{x}, t)\int \mathrm{d}\boldsymbol{x}' f(\boldsymbol{x}-\boldsymbol{x}')A(\boldsymbol{x}', t).$$

L'utilisation des relations

$$\{\psi_N(\boldsymbol{x}, t), \psi_N^+(\boldsymbol{x}', t)\} = \delta(\boldsymbol{x}-\boldsymbol{x}'),$$

$$[A(\boldsymbol{x}, t), A^+(\boldsymbol{x}', t)] = \beta(\boldsymbol{x}-\boldsymbol{x}') = \frac{1}{2(2\pi)^3}\int \mathrm{d}\boldsymbol{k}\,\frac{\exp[i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')]}{\omega_k},$$

conduit aussitôt à la formule

$$\langle 0|\{\psi_V(\boldsymbol{x}, t), \psi_V^+(\boldsymbol{x}', t)\}|0\rangle = \left(\frac{g}{\delta\nu}\right)^2 \delta(\boldsymbol{x}-\boldsymbol{x}')\int \mathrm{d}\boldsymbol{x}_1\,\mathrm{d}\boldsymbol{x}_1' f(\boldsymbol{x}-\boldsymbol{x}_1) f(\boldsymbol{x}'-\boldsymbol{x}_1')\beta(\boldsymbol{x}_1-\boldsymbol{x}_1'),$$

où le second membre peut encore s'écrire

$$\left(\frac{g}{2\pi\delta\nu}\right)^2 \delta(\boldsymbol{x}-\boldsymbol{x}') \cdot \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} \int_0^\infty \mathrm{d}k \, \frac{k f^2(\omega_k) \sin k|\boldsymbol{x}-\boldsymbol{x}'|}{\omega_k} .$$

La fonction

$$K(|\boldsymbol{x}|) = \frac{1}{|\boldsymbol{x}|} \int_0^\infty \mathrm{d}k \, \frac{k f^2(\omega_k) \sin k|\boldsymbol{x}|}{\omega_k}$$

étant finie et continue à l'origine si la fonction de coupure $f(\omega_k)$ décroit suffisamment vite à l'infini, on obtient simplement

$$\langle 0 | \{\psi_{\mathrm{V}}(\boldsymbol{x}, t), \psi_{\mathrm{V}}^+(\boldsymbol{x}', t)\} | 0 \rangle = \left(\frac{g}{2\pi\delta\nu}\right)^2 K(0) \, \delta(\boldsymbol{x}-\boldsymbol{x}') ,$$

avec

$$K(0) = \int_0^\infty \mathrm{d}k \, \frac{k^2 f^2(\omega_k)}{\omega_k} .$$

Remarquons que, d'après le calcul précédent, cette constante peut aussi s'écrire

$$K(0) = 4\pi^2 \langle 0 | \mathscr{A}(t) \, \mathscr{A}^+(t) | 0 \rangle,$$

où l'on a posé

$$\mathscr{A}(t) = \int \mathrm{d}\boldsymbol{x} f(\boldsymbol{x}) A(\boldsymbol{x}, t) .$$

Le cas que nous venons d'envisager présente de grandes analogies avec celui des champs vectoriels. En effet, si l'on considère la densité lagrangienne suivante:

$$-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{2}(1-c)(\partial_\mu A^\mu)^2 + \tfrac{1}{2} m^2 A_\mu A^\mu \qquad (F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu),$$

il en résulte pour A_4 le moment conjugué

$$\pi^4 = -(1-c) \, \partial_\mu A^\mu,$$

et la relation de commutation à temps constant

$$[\partial_4 A^4(\boldsymbol{x}), A_4(\boldsymbol{x}')] = -\frac{1}{1-c} \, \delta(\boldsymbol{x}-\boldsymbol{x}') .$$

Cette relation devient singulière quand $c \to 1$, tandis que simultanément le moment conjugué π^4 s'annule. Or on sait bien que dans ce cas la composante A_4 n'est plus indépendante des trois premières; la relation de commutation est en réalité (6)

$$[\partial_4 A^4(\boldsymbol{x}), A_4(\boldsymbol{x}')] = \frac{-i}{m^2} \Delta\, \delta(\boldsymbol{x} - \boldsymbol{x}')\,,$$

où la distribution figurant au second membre est plus singulière que la distribution usuelle δ.

Par ailleurs, A. S. WIGHTMAN (2) indique que dans une théorie où, par suite de la divergence de l'intégrale d'une fonction spectrale, un commutateur prend la forme $i \infty \delta(\boldsymbol{x} - \boldsymbol{y})$ celui-ci doit être remplacé par une distribution plus singulière.

Ce n'est pas ce qui se produit dans notre cas particulier où l'anticommutateur donné par ψ_V est encore proportionnel à δ. Quoi qu'il en soit l'étude précédente montre que le cas $Z = 0$ ne doit pas être considéré comme un cas limite mais bien comme un cas spécial, nécessitant une étude séparée, qui conduit à une théorie dont la structure est différente de celle correspondant au cas $Z \neq 0$

5. – Les deux définitions de la constante de renormalisation.

G. KÄLLÉN (7) donne, pour la constante de renormalisation de la particule V du modèle de Lee, les deux expressions équivalentes suivantes:

$$Z = 1 - \int_{\mu}^{\infty} \frac{\mathrm{d}\omega}{\omega}\, \Pi^*(\omega)\,, \tag{5.1}$$

$$\frac{1}{Z} = 1 + \int_{\mu}^{\infty} \frac{\mathrm{d}\omega}{\omega}\, \Pi(\omega)\,. \tag{5.2}$$

La première est celle que nous avons utilisée jusqu'ici. La second équivaut à la définition de H. LEHMANN (8)

$$\frac{1}{Z} = \int_{-\infty}^{\infty} \varrho(a)\, \mathrm{d}a\,,$$

(6) G. WENTZEL: *Quantum Theory of Fields* (1949), p. 75.

(7) G. KÄLLÉN: Rapport du C.E.R.N., Nov. 1955 (non publié). Voir aussi: H. LEHMANN, K. SYMANZIK and W. ZIMMERMANN: *Nuovo Cimento*, **2**, 435 (1955).

(8) H. LEHMANN: *Nuovo Cimento*, **11**, 342 (1954).

où $\varrho(a)$ est la fonction de poids qui fait connaître le propagateur renormalisé $S'_F(x)$, à partir du propagateur libre pour une particule V de masse a $S_F(x; a)$, par la formule

$$S'_F(x) = \int_{-\infty}^{\infty} \mathrm{d}a\, \varrho(a)\, S_F(x; a)\,, \qquad (x = (\boldsymbol{x}, t)). \tag{5.3}$$

Or s'il est aisément compréhensible qu'il soit possible d'obtenir la condition $Z = 0$ à partir de la formule (5.1), cela l'est moins avec la formule (5.2) car l'intégrale figurant au second membre doit être divergente, quelle que soit la rapidité de décroissance de la fonction de coupure, et sans que la fonction $\Pi(\omega)$ paraisse présenter quelque singularité.

Nous raisonnerons sur la formule de H. Lehmann et nous commencerons par déterminer la fonction de poids $\varrho(a)$.

Le propagateur libre de la particule V de masse a se détermine aisément; il a pour expression

$$S_F(x; a) = \exp[-iat]\theta(t)\,\delta(\boldsymbol{x}) = \frac{i}{(2\pi)^4}\int \mathrm{d}^4p\, \exp[-ip\cdot x]\frac{1}{p_0 - a + i\varepsilon}\,,$$
$$(p\cdot x = p_0 t - \boldsymbol{p}\cdot\boldsymbol{x}).$$

Introduisons la transformée de Fourier $S'_F(p)$ du propagateur renormalisé

$$S'_F(x) = \int \mathrm{d}^4p\, \exp[-ip\cdot x] S'_F(p)\,.$$

$S'_F(p)$ se détermine par la méthode habituelle en calculant les graphes de self-énergie de la particule V. On trouve

$$S'_F(p) = S'_F(p_0) = \frac{i}{(2\pi)^4}\,\frac{1}{[p_0 - m_V + i\varepsilon]\varphi^{(+)}(p_0)}\,,$$

où l'on a posé

$$\begin{cases} \varphi(z) = 1 + \dfrac{g^2}{4\pi^2}\,(z - m_V)\displaystyle\int_0^{\infty} \mathrm{d}k\, \frac{k^2 f^2(\omega_k)}{\omega_k[\omega_k + m_N - m_V]^2[\omega_k + m_N - z]}\,, \\ \varphi^{(+)}(p_0) = \varphi(p_0 + i\varepsilon)\,. \end{cases}$$

En égalant les parties imaginaires des deux membres de la relation (5.3) écrite avec les transformées de Fourier, on obtient immédiatement

$$\varrho(a) = \delta(a - m_V) + \frac{1}{\pi}\cdot\frac{1}{a - m_V}\cdot\frac{\operatorname{Im}\varphi^{(+)}(a)}{|\varphi^{(+)}(a)|^2}\,,$$

la relation de H. Lehmann donne alors pour expression de $1/Z$

$$\frac{1}{Z} = 1 + \frac{1}{\pi}\int_{m_N+\mu}^{\infty} \mathrm{d}a \, \frac{1}{a - m_V} \, \frac{\operatorname{Im} \varphi^{(+)}(a)}{|\varphi^{(+)}(a)|^2} \, .$$

Nous calculerons l'intégrale du second membre par la théorie des résidus, en suivant une méthode déjà utilisée par G. KÄLLÉN et W. PAULI [5].

La fonction $\varphi(z)$ est holomorphe dans tout le plan complexe à l'exception de la coupure $[m_N+\mu, +\infty]$ et ne possède pas de zéro à distance finie si la quantité

$$\varepsilon = \lim_{|z|\to\infty} \varphi(z) =$$

$$= 1 - \frac{g^2}{4\pi^2}\int_0^{\infty} \mathrm{d}k \, \frac{k^2 f^2(\omega_k)}{\omega_k [\omega_k + m_N - m_V]} \, ,$$

n'est pas négative.

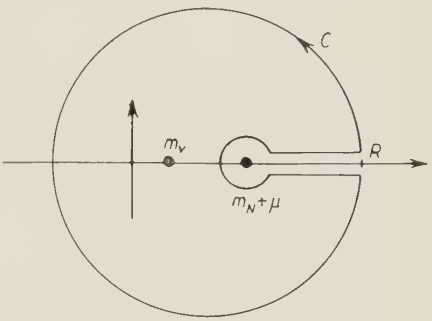

Fig. 1.

L'application du théorème des résidus au contour Γ ci-contre donne alors facilement la formule

$$\int_C \frac{\mathrm{d}a}{(a - m_V)\varphi(a)} - 2i\int_{m_N+\mu}^{R} \mathrm{d}a \, \frac{\operatorname{Im} \varphi^{(+)}(a)}{(a - m_V)|\varphi^{(+)}(a)|^2} = \int_\Gamma \frac{\mathrm{d}a}{(a - m_V)\varphi(a)} = 2\pi i \, ,$$

d'où il s'ensuit

$$\frac{1}{Z} = \lim_{R\to\infty} \frac{1}{2\pi i}\int_C \frac{\mathrm{d}a}{(a - m_V)\varphi(a)} \, .$$

Si ε est positif on trouve immédiatement la limite de cette intégrale et l'on obtient $Z = \varepsilon$. C'est bien la valeur utilisée à la Section 2.

Si ε est nul, l'intégrale ci-dessus diverge. Bien que l'on n'ait pas trouvé de preuve générale il est possible de s'en convaincre en considérant le cas particulier défini par les données suivantes:

$$m_V = \mu = 0 \, , \qquad m_N = m \, , \qquad f(\omega_k) = \sqrt{\frac{m}{k}} \, .$$

Remarquons d'abord, qu'en général, $\varphi(z)$ peut s'écrire

$$\varphi(z) = \varepsilon + \alpha\int_0^{\infty} \mathrm{d}k \, \frac{k^2 f^2(\omega_k)}{\omega_k [\omega_k + m_N - m_V][\omega_k + m_N - z]} \, ,$$

où l'on a posé $\alpha = g^2/4\pi^2$ ($= 1$ si $\varepsilon = 0$ dans le cas particulier choisi). On a alors, dans le cas particulier envisagé

$$\varphi(z) = \varepsilon + \alpha m \int_0^\infty \frac{\mathrm{d}k}{(k+m)(k+m-z)} = \varepsilon + \frac{\alpha m}{z} \log \frac{m}{m-z},$$

où le logarithme est réel quand $z < m$.

Quand $\varepsilon = 0$, en centrant le cercle C au point m et en posant $m - z = R \exp[i(\theta - \pi)]$ on obtient enfin

$$\int_C = \frac{1}{\alpha m} \int_C \frac{\mathrm{d}z}{\log m/(m-z)} = \frac{Ri}{\alpha m} \int_0^{2\pi} \frac{\exp[i\theta]\,\mathrm{d}\theta}{\log m/R + i\pi - i\theta},$$

et cette dernière expression tend bien vers l'infini quand le rayon R du cercle croît indéfiniment.

Ainsi le formule (5.2) est elle aussi correcte dans le cas $Z = 0$. D'une manière plus physique, la divergence de l'intégrale de $\varrho(a)$ peut se comprendre en remarquant que, par exemple dans le cas particulier ci-dessus, cette fonction possède un pic de hauteur constante dont l'abscisse et la largeur croissent indéfiniment quand ε tend vers zéro.

6. – Conclusion.

Les pages qui précèdent nous montrent, au moins pour le cas particulier envisagé, que si dans une théorie à couplage de Yukawa il y a annulation de la constante de renormalisation de l'un des champs, la particule correspondante peut se représenter comme un état lié formé à partir des autres particules intervenant dans le couplage, la théorie étant alors équivalente à une théorie à couplage de Fermi mettant en jeu un nombre moindre de particules.

Remarquons que, dans le cas du modèle de Lee précédemment traité, la la constante Z ne s'annule pas identiquement mais seulement quand une relation définie existe entre la fonction de coupure et les paramètres définissant le modèle.

D'après G. Källén, il se pourrait que l'électrodynamique possède une constante de renormalisation identiquement nulle (quelles que soient les valeurs des paramètres: charge et masse). On ne connait pas actuellement de modèle calculable présentant cette particularité et nous ne savons pas si notre conclusion est encore valable dans ce cas.

Si nous pouvons, dans les deux cas, considérer comme nous l'avons fait, cette annulation comme une contrainte alors le photon doit être un état lié résultant de couplages de Fermi et ayant, selon le cas, une charge déterminée ou arbitraire.

RIASSUNTO (*)

Si mostra su un esempio risolubile che una teoria ad accoppiamento di Fermi è equivalente ad una teoria ad accoppiamento di Yukawa nella quale la costante di rinormalizzazione della particella intermedia sia nullo. L'annullamento di questa costante viene successivamente interpretata in un formalismo lagrangiano come una limitazione imposta ai campi. Si studia infine la singolarità delle relazioni di commutazione.

(*) *Traduzione a cura della Redazione.*

Cellular Space-time and Quantum Field Theory.

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Summary. — A simple cellular model of space-time is introduced which incorporates a fundamental length in a natural way. The usual concept of field quantities as functions of geometrical points is rejected in favour of the concept of field quantities as functionals of elementary cells. Consequently the usual field equations in the form of partial differential equations are replaced by corresponding partial difference equations. Subsequent quantization yields an S-matrix which is not vitiated by the presence of serious divergences. The basic space-time structure and the S-matrix (which does not involve derivative coupling) are shown to be integral Lorentz covariant. The semi-convergence or asymptotic behaviour of the S-matrix series is established, and a meaningful Borel-sum of the « gross » S-matrix series is presented.

1. – Introduction.

In spite of the formal successes of renormalized field theories, the procedure of subtraction of infinities is mathematically dubious. Besides, renormalization is not universally applicable and fails in the cases of derivative couplings. Also the approach of renormalization necessarily fails to shed any light on the connection between elementary masses and self-interactions.

Usual field theories are based on the exact measurability of the field operators defined at precise space-time points, and this is questionable [1]. Furthermore, it is known [2-5] that additional boundary divergences are intro-

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duced into the theory if the space-time region where the field quantities are defined has sharp spatio-temporal boundaries. Apart from these theoretical considerations, the success of the Pauli-Chew model ([6,7]) in fitting the experimental curves of meson-nucleon processes, and other recent experiments ([8,9]), definitely indicate the non-local character of elementary interactions.

From these considerations it seems that the most plausible step will be to introduce a fundamental length ([10-13]) to overcome the difficulties associated with the present day field theories. Attempts were made to achieve this end either by introducing structure in space-time ([14-20]) or by introducing non-local interactions ([21-31]) or non-local fields ([32-36]). But, in spite of these endeavours,

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([23]) R. P. Feynman: *Phys. Rev.*, **74**, 939 (1948).

([24]) R. E. Peierls and H. McManus: *Phys. Rev.*, **70**, 795 (1948).

([25]) H. McManus: *Proc. Roy. Soc.*, A **195**, 323 (1949).

([26]) C. Bloch: *Kgl. Dan. Vidensk. Selsk., Mat.-Fys. Medd.*, **26**, no. 1 (1950); **27**, no. 8 (1952).

([27]) J. Rayski: *Phil. Mag.*, **42**, 1289 (1951).

([28]) P. Kristensen and C. Møller: *Kgl. Dan. Vidensk. Selsk., Mat.-Fys. Medd.*, **27**, no. 7 (1952).

([29]) M. Chrétien and R. E. Peierls: *Proc. Roy. Soc.*, A **223**, 468 (1954).

([30]) E. Arnous and W. Heitler: *Nuovo Cimento*, **11**, 443 (1959).

([31]) E. Arnous, W. Heitler and Y. Takahashi: *Nuovo Cimento* **16**, 671 (1960).

([32]) M. A. Markov: *Žurn. Èksp. Teor. Fiz.*, **10**, 1311 (1940).

([33]) H. Yukawa: *Phys. Rev.*, **76**, 300 (1949); **77**, 219, 849 (1950); **80**, 1047 (1950).

([34]) A. Pais and G. E. Uhlenbeck: *Phys. Rev.*, **79**, 145 (1950).

([35]) O. Hara, T. Marumori, Y. Ohnuki and H. Shimodaira: *Progr. Theor. Phys.*, **12**, 177 (1954).

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there does not exist a fully fledged quantum theory of fields incorporating a fundamental length satisfactorily. Our aim is to achieve this. The idea is to replace the continuously infinite degrees of freedom of the field by denumerably infinite degrees of freedom with the introduction of space-time cells involving a fundamental length. This in turn demands replacement of partial differential field equations by partial difference equations. (Partial difference field equations have been used previously with different motivations ([37-41])). Next a consistent quantization scheme is developed following closely the analogy suggested by the existing theory. As could be anticipated, the resulting S-matrix has no serious divergences.

2. – Preliminaries of the calculus of finite differences ([42-44]).

Let x be a discrete variable ($x = 0, \pm 1, \pm 2, ..., \pm\infty$), and $f(x)$ be a function of x (with $|f(x)| < \infty$); then various finite differences may be defined as follows:

$$
(2.1)\quad \begin{cases} \text{Right difference} & \Delta f(x) = f(x+1) - f(x)\,, \\ \text{Left difference} & \Delta' f(x) = f(x) - f(x-1)\,, \\ \text{Symmetric difference} & \bar{\Delta} f(x) = \tfrac{1}{2}[f(x+1) - f(x-1)]\,, \\ \text{Forward operation} & Ef(x) = f(x+1)\,. \end{cases}
$$

Evidently $\Delta' = \Delta E^{-1}$, $\bar{\Delta} = \frac{1}{2}(\Delta + \Delta')$, $E = 1 + \Delta$, and for a continuous function $\Delta = \exp[\mathrm{d}/\mathrm{d}x] - 1$, (Lagrange's relation), so that difference equations corre-

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([38]) R. Courant, K. Friedrichs and H. Lewy: *Math. Ann.*, **100**, 32 (1928).

([39]) B. T. Darling: *Phys. Rev.*, **80**, 460 (1951); **92**, 1547 (1953); L. I. Schiff: *Phys. Rev.*, **92**, 766 (1953).

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([41]) E. W. Montroll: *Amer. Math. Month.*, **61**, 7 (1954).

([42]) G. Boole: *Calculus of Finite Differences*, 4th. ed. (New York, 1958).

([43]) C. Jordan: *Calculus of Finite Differences*, 2nd ed. (New York, 1950).

([44]) N. E. Nörlung: *Vorlesungen über Differenzenrechnung* (Ann Arbor, 1945).

spond to infinite order differential equations. It follows from (2.1) that

$$
(2.2)\quad
\begin{cases}
\Delta \sum\limits_{n} f_{(n)}(x) = \sum\limits_{n} \Delta f_{(n)}(x)\,, \\
\Delta \prod\limits_{n} f_{n}(x) = \prod\limits_{n} f_{n}(x) \left[\sum\limits_{i} \Delta f_{i}(x) \Big/ f_{i}(x) + \sum\limits_{i} \sum\limits_{j} \Delta f_{i}(x)\, \Delta f_{j}(x) \Big/ f_{i}(x)\, f_{j}(x) + \ldots \right], \\
\text{[Special case:} \\
\Delta [f_{(1)}(x)\, f_{(2)}(x)] = f_{(2)}(x)\, \Delta f_{(1)}(x) + f_{(1)}(x)\, \Delta f_{(2)}(x) + \Delta f_{(1)}(x)\, \Delta f_{(2)}(x) = \\
\qquad\qquad + f_{2}(x-1)\, \Delta f_{1}(x) + f_{1}(x)\, \Delta f_{2}(x) + f_{2}(x)\, \Delta f_{1}(x) + f_{1}(x-1)\, \Delta f_{2}(x) \big], \\
E^{m} \prod\limits_{n} f_{(n)}(x) = \prod\limits_{n} [E^{m} f_{(n)}(x)]\,.
\end{cases}
$$

Some simple examples:

$$
(2.3)\quad
\begin{cases}
\bar{\Delta} \exp[-ikx] = -i \sin k \exp[-ikx]\,, \\
\Delta\Delta' \exp[-ikx] = -4 \sin^{2}(k/2) \exp[-ikx]\,, \qquad (\Delta' \sin kx)_{x=0} = \sin k\,.
\end{cases}
$$

Successive differences:

$$
(2.4)\qquad \Delta^{n} f(x) = (E-1)^{n} f(x) = \sum_{m=0}^{n} (-1)^{m} \binom{n}{m} f(x+n-m)\,.
$$

Expansion of a function:

$$
(2.5)\qquad f(x+n) = (1+\Delta)^{n} f(x) = \sum_{m=0}^{n} \binom{n}{m} \Delta^{m} f(x)\,.
$$

Indefinite summation (provided the sum exists):

$$
(2.6)\qquad -\Delta^{-1} f(x) = (1-E)^{-1} f(x) = \sum_{x} f(x+n)\,.
$$

Summation by parts:

$$
(2.7)\qquad \Delta^{-1} [f_{1}(x)\, f_{2}(x)] = f_{(1}(x)\, \Delta^{-1} f_{2)}(x) - \Delta^{-1} [\{\Delta f_{(1)}(x)\} \{\Delta^{-1} f_{2}(x+1)\}]\,.
$$

Partial differences:

$$
(2.8)\qquad \Delta_{\alpha} f(x_{1}, \ldots, x_{\alpha}, \ldots) = f(x_{1}, \ldots, x_{\alpha}+1, \ldots) - f(x_{1}, \ldots, x_{\alpha}, \ldots)\,.
$$

Also, $\Delta'_{\alpha} = \Delta_{\alpha} E_{\alpha}^{-1}$, $\bar{\Delta}_{\alpha} = \tfrac{1}{2}(\Delta_{\alpha} + \Delta'_{\alpha})$, $E_{\alpha} = 1 + \Delta_{\alpha}$, and these operators commute among themselves.

3. – The « truncated » Fourier transform.

Let $f(x)$ be a bounded step function (having constant value $f(x_n)$ in the interval $x_n \geqslant x \geqslant x_n - 1$; x_n being integers) defined over $\infty \geqslant x \geqslant -\infty$, and having a denumerably infinite number of discontinuities. This function will possess a unique Fourier transform defined by

$$f(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} g(k) \exp[ikx]\,\mathrm{d}k\,, \tag{3.1}$$

so that the inverse relation will give

$$g(k) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} f(x) \exp[-ikx]\,\mathrm{d}x = \frac{1}{(2\pi)^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} f(x_n) \int_{x_n-1}^{x_n} \exp[-ikx]\,\mathrm{d}x = \tag{3.2}$$

$$= \frac{1}{(2\pi)^{\frac{1}{2}}} \left\{\frac{1-\exp[ik]}{-ik}\right\} \sum_{n'=-\infty}^{\infty} f(x_{n'}) \exp[-ikx_{n'}]\,.$$

We shall define the « truncated » Fourier transform of the original function by

$$g'(k) = g(k) \left\{\frac{ik}{\exp[ik]-1}\right\} = \frac{1}{(2\pi)^{\frac{1}{2}}} \sum_{n'=-\infty}^{\infty} f(x_{n'}) \exp[-ikx_{n'}]\,,$$

so that

$$f(x_n) = \sum_{n'=-\infty}^{\infty} f(x_{n'})\,\delta_{nn'} = \frac{1}{2\pi} \sum_{n'=-\infty}^{\infty} f(x_{n'}) \int_{-\pi}^{\pi} \exp[ik(x_n - x_{n'})]\,\mathrm{d}k = \tag{3.3}$$

$$= \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\pi}^{\pi} g'(k) \exp[ikx_n]\,\mathrm{d}k\,.$$

This « truncated » transform will reproduce the function $f(x)$ correctly only at the lattice or discrete points x_n. In the subsequent investigation our main interest will lie in the values of the function at lattice points and we shall use this relationship extensively ([45]).

([45]) One can use the complete Fourier transform, but it happens in this work that the integrals always boil down to $\int_{-\pi}^{\pi}(\ldots)\,\mathrm{d}k$ through the theorem

$$\int_{-\infty}^{\infty} \frac{f(\sin k)}{k} \sin k\,\mathrm{d}k = \int_{-\pi}^{\pi} f\left(\sin\frac{k}{2}\right)\frac{\mathrm{d}k}{2}\,.$$

Now, we shall make a slight digression, to introduce an improper function (for subsequent use) in the following operations:

$$\left| \int_b^a \bar{\delta}(k)\,\mathrm{d}k = \lim_{N\to\infty} \frac{1}{2\pi} \int_b^a \Big[\sum_{n=-N}^{N} \exp\,[ikx_n] \Big]\,\mathrm{d}k = 1\,, \right.$$

(3.4)

$$\left| \int_b^a f(k)\,\bar{\delta}(k)\,\mathrm{d}k = f(0)\,, \right.$$

provided $2\pi \geqslant a > 0$, $0 > b \geqslant -2\pi$, and $f(k)$ is analytic in the domain $a \geqslant k \geqslant b$. Thus $\bar{\delta}(k)$ is equivalent to the delta-function $\delta(k)$ in $2\pi \geqslant k \geqslant -2\pi$ and we shall omit the bar hereinafter. Also the subscript n in x_n will be dropped, with the understanding that x itself will stand for an integral number. (Loosely speaking $\bar{\delta}(k) \sim \sum_n \delta(k + 2\pi n)$, so that according to the cellular model at very high energies some new channels (so called « umklapprozesse ») will be opened which have no analogue in the continuous model.)

4. – The lagrangian formalism.

The units are chosen to be such that $l = \hbar = c = 1$, so that all other physical quantities are expressed as pure numbers. (This is a natural choice, because nature provides three basic dimensional constants l, $\hbar$, c through her limitations in measurability in ordinary space-time and in phase-space, also through her inability to propagate action instantaneously. All other physical constants should be explained and expressed in terms of these three constants, while the magnitudes of these three should elude any theoretical explanation.)

The Latin, and roman bold face Latin, indices will stand for space-time and space components respectively, whereas Greek indices will denote spinor components. The summation convention is followed except when an index occurs inside an argument (like $\Delta_a \varphi(E_a^{-1}x)$). The metric tensor is taken to be

$$[\eta_{ab}] = [\eta^{ab}] \equiv \mathrm{diag}\,[-1, -1, -1, +1].$$

One typical hypercubic space-time cell (one lattice point having 16 immediate neighbours) is characterized by quadruple integers x^a (in short x), and any typical field quantity as a functional of cell (or a function of quadruple integers) is denoted by $\underset{(b)}{\varphi}(x)$ (where the subscript (b) stands for tensor or spinor components). Then the variational principle can be stated as

$$\delta A = \delta \sum_{\text{space-time cells } (\Omega)} L\{x, \underset{(b)}{\varphi}(x), \Delta'_a \underset{(b)}{\varphi}(x)\} = 0\,, \tag{4.1}$$

together with

$$\delta \underset{(b)}{\varphi}[\sigma_{-1}(\Omega)] = 0 ,$$

where Ω denotes a space-time region filled by hypercubic cells and $\sigma(\Omega)$ is the hyperplane constructed from outward surfaces of boundary cells, and $\sigma_{-1}(\Omega)$ is the hyperplane constructed from the « inward » surfaces of boundary cells. (4.1) yields the following field equation

$$\Delta_a \frac{\partial L}{\partial \Delta'_a \underset{(b)}{\varphi}(x)} - \frac{\partial L}{\partial \underset{(b)}{\varphi}(x)} = 0 . \tag{4.2}$$

The invariance of the Lagrangian under different groups of transformations, and the corresponding conservation laws, are stated below:

i) L is invariant under the group of translations characterized by $x' = E^{a}{}_{b} x$; so that the canonical energy-momentum tensor of the field is conserved as $\Delta_b T^{ab} = 0$, where

$$T_{ab} = \Delta_a \underset{(c)}{\varphi}(E_b^{-1}x) \frac{\partial L}{\partial \Delta'^{b} \underset{(c)}{\varphi}(x)} - \eta_{ab} L + \dots . \tag{4.3}$$

ii) L is invariant under gauge transformation of the first kind, *viz.* $\underset{(a)}{\varphi'} = \underset{(a)}{\varphi} \exp[i\varepsilon]$, $\underset{(a)}{\varphi'^{*}} = \underset{(a)}{\varphi^{*}} \exp[-i\varepsilon]$; so that the charge-current vector of the field is conserved as $\Delta_a j^a = 0$, where

$$j^a = ie \left[\frac{\partial L}{\partial \Delta'_a \underset{(b)}{\varphi}(x)} \underset{(b)}{\varphi}(E_a^{-1}x) - \underset{(b)}{\varphi^{*}}(E_a{}^{-1}x) \frac{\partial L}{\partial \Delta'_a \underset{(b)}{\varphi^{*}}(x)} \right] . \tag{4.4}$$

iii) L is invariant under groups of space-reflection, time-reversal, and charge-conjugation (we are not concerned with the weak interactions).

iv) Integral Lorentz-transformations ([16,17]):

A homogeneous Lorentz-transformation is characterized by

$$x' = \mathscr{L} x , \tag{4.5a}$$

$$\tilde{\mathscr{L}} \eta \mathscr{L} = \eta . \tag{4.5b}$$

The integral Lorentz-group is an infinite sub-group of the continuous homogeneous Lorentz-group, an element of which maps the discrete space-time onto itself, *i.e.*, leaves the hypercubic lattice-structure invariant. The possibility of having non-trivial rotations in space-time (trivial rotations being the multiples of « right-angle ») which leave the lattice-structure intact betrays our ordinary common sense, and is possible solely due to the indefinite metric and four-dimensionality of space-time. In an integral Lorentz-transformation, all the components $\mathscr{L}^{a}{}_{b}$ should necessarily be real integers, and thus by (4.5*b*) the problem of determining all integral Lorentz-transformations reduces to the

solution of ten quadratic diophantine equations (*e.g.* $\pm\alpha^2 \quad \beta^2 \pm \gamma^2 \pm \delta^2 = \pm 1, 0$) in sixteen unknown integers. Typical integral Lorentz-transformations are given by

$$(4.6)\qquad \begin{cases} [\underset{\text{I}}{\mathscr{L}}] = \begin{bmatrix} 2 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{bmatrix}, \qquad [\underset{\text{II}}{\mathscr{L}}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \\ [\underset{\text{III}}{\mathscr{L}}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad [\underset{\text{IV}}{\mathscr{L}}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{cases}$$

Incidentally, it can be proved that the operation $\underset{\text{I}}{\mathscr{L}}$, $\underset{\text{II}}{\mathscr{L}}$, $\underset{\text{III}}{\mathscr{L}}$, $\underset{\text{IV}}{\mathscr{L}}$ generate the group of all integral Lorentz-transformations, and that this infinite discrete group can be abstractly defined by the generating relations

$$(4.7)\qquad \underset{\text{I}}{\mathscr{L}}^2 = \underset{\text{II}}{\mathscr{L}}^2 = \underset{\text{III}}{\mathscr{L}}^2 = \underset{\text{IV}}{\mathscr{L}}^2 = (\underset{\text{I}}{\mathscr{L}}\,\underset{\text{III}}{\mathscr{L}})^2 = (\underset{\text{I}}{\mathscr{L}}\,\underset{\text{IV}}{\mathscr{L}})^2 = (\underset{\text{II}}{\mathscr{L}}\,\underset{\text{IV}}{\mathscr{L}})^2 = \\ = (\underset{\text{I}}{\mathscr{L}}\,\underset{\text{II}}{\mathscr{L}})^4 = (\underset{\text{II}}{\mathscr{L}}\,\underset{\text{III}}{\mathscr{L}})^4 = (\underset{\text{III}}{\mathscr{L}}\,\underset{\text{IV}}{\mathscr{L}})^3 = 1\,.$$

Though under the integral Lorentz-transformation the basic cellular space-time structure remains unaltered, the covariance of the partial difference equations cannot be claimed (in spite of the fact that the difference equations treat space and time on the same footing). Therefore, the usual angular momentum conservations of the free fields cannot be derived. But the situation is not quite hopeless, because of the fact that the interaction Lagrangian for quantum electrodynamics $e\,\bar{\psi}(x)\gamma^a A_a(x)\psi(x)$ (or in fact for any non-derivative coupling case) is integral Lorentz-invariant. Therefore, the corresponding S-matrix (which alone relates the physical observations to the theory) is invariant under integral Lorentz-transformations.

5. – The non-singular Green's functions.

We shall define various boson-propagators in the following way

$$(5.1)\qquad \begin{cases} \underset{(a)}{\Delta}(xx') = \dfrac{1}{(2\pi)^4}\displaystyle\int_{-\pi}^{\pi}\!\!\int\!\!\int_{C_{(a)}} \frac{\exp\left[-ik_b(x^b - x'^b)\right]\mathrm{d}^4k}{-4\eta^{cd}\sin k_c/2\,\sin k_d/2 + \mu^2}, \\ \underset{(a)}{D}(xx') = \lim_{\mu\to 0}\underset{(a)}{\Delta}(xx')\,, \end{cases}$$

where the different contours $C_{(a)}$ in the complex k_4-plane are shown in Fig. 1, two singularities being situated at

$$2 \sin k_4/2 = \pm \sqrt{[4(\sin^2 k_1/2 + \sin^2 k_2/2 + \sin^2 k_3/2) + \mu^2]} = \pm |\omega(\boldsymbol{k})| .$$

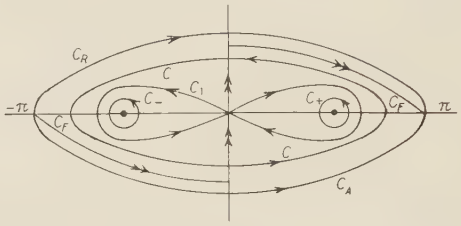


Fig. 1.

Integrals with closed contours are called homogeneous, and with open contours are inhomogeneous. They satisfy, from (5.1) and (2.3),

$$(5.2)\quad \begin{cases} [\eta^{ab}\Delta_a \Delta'_b + \mu^2] \underset{(\mathrm{inh})}{\Delta}(xx') = \delta_{xx'} , \\ [\eta^{\ b}\Delta_a \Delta'_b + \mu^2] \underset{(\mathrm{hom})}{\Delta}(xx') = 0 , \\ |\underset{(a)}{\Delta}(xx')| < \infty , \\ \Delta(xx') = -\dfrac{1}{(2\pi)^3} \displaystyle\int_{-\pi}^{\pi}\int\int\int \exp\left[-ik_{\mathbf{a}}(x^{\mathbf{a}} - x'^{\mathbf{a}})\right] \frac{\sin k_0(x^4 - x'^4)}{\sin k_0} \, \mathrm{d}^3\boldsymbol{k} , \end{cases}$$

(where $2 \sin k_0/2 = |\omega(\boldsymbol{k})|$),

$$\lim_{x^4 = x'^4 \to 0} [\Delta'_4 \Delta(xx')] = -\delta_{\mathbf{xx}'} , \qquad \lim_{x^4 \to x'^4} [\Delta(xx')] = 0 .$$

Definitions for fermion-propagators are given below,

$$(5.3)\quad \underset{(a)}{S}_{\alpha\beta}(xx') = -\frac{1}{(2\pi)^4} \int_{-\pi}^{\pi}\int\int\int_{C_{(a)}} \frac{[\gamma^a \sin p_a + mI]_{\alpha\beta}}{-\eta^{cd} \sin p_c \sin p_d + m^2} \exp\left[-ip_b(x^b - x'^b)\right] \mathrm{d}^4\boldsymbol{p} ,$$

so that

$$(5.4)\quad \begin{cases} [i\gamma^a \overline{\Delta}_a - mI]_{\alpha\gamma} \underset{(\mathrm{inh})}{S}_{\gamma\beta}(xx') = \delta_{\alpha\beta}\delta_{xx'} , \\ [i\gamma^a \overline{\Delta}_a - mI]_{\alpha\gamma} \underset{(\mathrm{hom})}{S}_{\gamma\beta}(xx') = 0 . \end{cases}$$

It can be easily seen that the homogeneous causal propagators $\underset{(\mathrm{hom})}{\Delta}(xx')$, $\underset{(\mathrm{hom})}{S}_{\alpha\beta}(xx')$ are different from zero in general only if $|\mu| < 2$, $|m| < 1$ (otherwise the poles in the complex k_4-plane will move out of the domain of integ-

ration). These limits provide the natural upper bound for the bare elementary boson and fermion masses. The upper limits of the frequency and wave number of a free field are π.

6. – Quantization of free fields.

i) Free scalar (or pseudoscalar) field: The Lagrangian of the free scalar (or pseudo-scalar) field is chosen to be

$$L = \tfrac{1}{2}[\eta^{ab}\,\Delta'_a\varphi(x)\,\Delta'_b\varphi(x) - \mu^2\varphi^2(x)]\,. \tag{6.1}$$

The variational principle (4.2) yields the field equation

$$[\eta^{ab}\,\Delta_a\,\Delta'_b + \mu^2]\,\varphi(x) = 0\,. \tag{6.2}$$

The total energy and momentum of the free field (from (4.3), (6.1)) are given by

$$\begin{cases} \mathrm{H}(0) = \sum\limits_{\text{space-cells}} T_{44}(\boldsymbol{x}, 0) = \tfrac{1}{2} \sum\limits_{\text{space-cells}} [\Delta'_a\varphi(\boldsymbol{x}, 0)\,\Delta'_a\varphi(\boldsymbol{x}, 0) + \\ \qquad\qquad + \Delta'_4\varphi(\boldsymbol{x}, 0)\,\Delta'_4\Delta_4\varphi(\boldsymbol{x}, 0) + \mu^2\varphi^2(\boldsymbol{x}, 0)]\,, \\ P_{\mathbf{a}}(0) = \sum\limits_{\text{space-cells}} T_{\mathbf{a}4}(\boldsymbol{x}, 0) = \sum\limits_{\text{space-cells}} [\Delta_{\mathbf{a}}\varphi(\boldsymbol{x}, -1)\,\Delta'_4\varphi(\boldsymbol{x}, 0)] + \ldots]\,. \end{cases} \tag{6.3}$$

Now, we shall introduce the truncated Fourier transform of the field (which is a function of quadruple integers x^a) by

$$\varphi(x) = \frac{1}{(2\pi)^2}\int\!\!\int\!\!\int\!\!\int_{-\pi}^{\pi} f(k)\,\exp\,[-ik_a x^a]\,\mathrm{d}^4k\,. \tag{6.4}$$

But as $\varphi(x)$ satisfies (6.2) we can write

$$\varphi(x) = \frac{1}{(2\pi)^2}\int\!\!\int\!\!\int\!\!\int_{-\pi}^{\pi} \int_C \frac{b(k)\,\exp\,[-ik_a x^a]\,\mathrm{d}^4k}{-4\eta^{bc}\,\sin\,k_b/2\,\sin\,k_c/2 + \mu^2} = \tag{6.5}$$

$$= \frac{1}{(2\pi)^3}\int\!\!\int\!\!\int_{-\pi}^{\pi} \frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{2\,|\omega(\boldsymbol{k})|\,\sqrt{[1-\omega^2(\boldsymbol{k})/4]}}}\,[a(\boldsymbol{k})\,\exp\,[-ik_a x^a] + a^*(\boldsymbol{k})\,\exp\,[ik_a x^a]] = \varphi^{(-)}(x) + \varphi^{(+)}(x)\,,$$

where $b(k)$ is assumed to be analytic in the complex k_4-plane and satisfies the reality condition $b^*(-k) = b(k)$ and

$$a(\boldsymbol{k}) = -ib(\boldsymbol{k}, k_0)\,\sqrt{\frac{\pi}{|\omega(\boldsymbol{k})|\,\sqrt{(1-\omega^2/4)}}} \qquad \text{and} \qquad k_a x^a = k_{\mathbf{a}}\,x^{\mathbf{a}} + k_0 x^4\,,$$

(k_0 was defined in Section 5).

Now we shall impose the quantum conditions

$$\begin{cases} [a(\boldsymbol{k}),\ a\ (\boldsymbol{k}')]_- = [a^*(\boldsymbol{k}),\ a^*(\boldsymbol{k}')]_- = 0\,, \\ [a(\boldsymbol{k}),\ a^*(\boldsymbol{k}')]_- = \delta(\boldsymbol{k} - \boldsymbol{k}')\,, \end{cases} \tag{6.6}$$

and reinterpret previous equations of Sections **4** to **6** as operational equations. From (6.5) and (6.6) one obtains

$$\begin{cases} [\varphi(x), \quad \varphi(x')]_- \quad = i\Delta(xx')\,, \\ [\varphi(\boldsymbol{x}, x^4),\ \varphi(\boldsymbol{x}', x^4)]_- = 0, \\ [\pi(\boldsymbol{x}, 0),\ \varphi(\boldsymbol{x}', 0)]_- = [\Delta_4'\varphi(\boldsymbol{x}, 0),\ \varphi(\boldsymbol{x}', 0)]_- = -i\delta_{\boldsymbol{x}\boldsymbol{x}'}\,, \\ [\varphi^{(+)}(x),\ \varphi^{(+)}(x')]_- = [\varphi^{(-)}(x),\ \varphi^{(-)}(x')]_- = 0\,, \\ [\varphi^{(-)}(x),\ \varphi^{(+)}(x')]_- = i\Delta_+(xx'), \\ [\varphi^{(+)}(x),\ \varphi^{(-)}(x')]_- = i\Delta_-(xx')\,. \end{cases} \tag{6.7}$$

The vacuum is defined as $a(\boldsymbol{k})|\Psi_0\rangle = 0$. The commutation relation $[\pi(\boldsymbol{x}, 0),\ \varphi(\boldsymbol{x}', 0)]_- = -i\delta_{\boldsymbol{x}\boldsymbol{x}'}$ (which corresponds to the classical Poisson bracket relation $[P_a,\ Q_b]_{\text{P.B.}} = \delta_{ab}$), was first written by Heisenberg and Pauli ([46]).

The energy and momentum of the free field become (from (6.3) and (6.5))

$$\begin{cases} \mathrm{H}(0) = \int\limits_{-\pi}^{\pi}\!\!\int\!\!\int \mathrm{d}^3\boldsymbol{k}\ [N(\boldsymbol{k}) + \tfrac{1}{2}]\ \omega(\boldsymbol{k})\ \sqrt{1 - \omega^2(\boldsymbol{k})/4}\,, \\ P_{\mathbf{a}}(0) = \int\limits_{\pi}^{\pi}\!\!\int\!\!\int \mathrm{d}^3\boldsymbol{k} N(\boldsymbol{k})\ [\sin k_{\mathbf{a}}], \end{cases} \tag{6.8}$$

where the number operator is as usual $N(\boldsymbol{k}) = a^*(\boldsymbol{k})a(\boldsymbol{k})$.

ii) Free electromagnetic field: The Lagrangian of the free electromagnetic field is

$$L = -\tfrac{1}{2}[\eta^{ab}\Delta_a' A_c(x)\,\Delta_b' A^c(x)]\,, \tag{6.9}$$

and the corresponding field equations are

$$\eta^{ab}\Delta_a\Delta_b' A_c(x) = 0\,. \tag{6.10}$$

([46]) W. Heisenberg and W. Pauli: *Zeits. Phys.*, **56**, 1 (1929).

The subsidiary condition to be imposed on the state vector is

$$\Delta'_a A^a(x)\,|\Psi\rangle = 0\,. \tag{6.11}$$

The truncated Fourier decomposition of the field is furnished by

$$\left\{\begin{aligned} A_a(x) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int\!\!\int\!\!\int_{-\pi}^{\pi} \frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{2\,|\underset{(0)}{\omega}(\boldsymbol{k})|\,\sqrt{(1-\underset{(0)}{\omega}^2/4)}}}\,[a_a(\boldsymbol{k})\exp[-ik_a x^a] + \\ &\qquad + a^*_a(\boldsymbol{k})\exp[ik_a x^a]] = \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int\!\!\int\!\!\int_{-\pi}^{\pi} \frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{2\,|\underset{(0)}{\omega}(\boldsymbol{k})|\,\sqrt{(1-\underset{(0)}{\omega}^2/4)}}}\sum_{(\lambda)=1}^{4}\varepsilon_a^{(\lambda)}(\boldsymbol{k})[a_{(\lambda)}(\boldsymbol{k})\cdot \\ &\qquad \cdot\exp[-ik\,x^a] + a^*_{(\lambda)}(\boldsymbol{k})\exp[ik_a x^a]] = \\ &= A_a^{(-)}(x) + A_a^{(+)}(x)\,, \end{aligned}\right. \tag{6.12}$$

where $\underset{(0)}{\omega}(\boldsymbol{k}) = \lim_{\mu\to 0}\omega(\boldsymbol{k})$, and $\varepsilon_a^{(\lambda)}(\boldsymbol{k})$'s are unit polarization vectors which can be so chosen to form an orthonormal tetrad satisfying

$$\varepsilon_a^{(\lambda)}\varepsilon^{(\lambda')a} = \eta^{(\lambda\lambda')}\,, \qquad \sum_{(\lambda)}\varepsilon_a^{(\lambda)}\varepsilon_b^{(\lambda)} = \eta_{ab}\,. \tag{6.1^c}$$

The quantum conditions to be imposed are

$$\left\{\begin{aligned} &[a_b(\boldsymbol{k}),\, a_c(\boldsymbol{k}')]_- = [a^*_b(\boldsymbol{k}),\, a^*_c(\boldsymbol{k}')]_- = 0\,, \\ &[a_b(\boldsymbol{k}),\, a^*_c(\boldsymbol{k}')]_- = -\eta_{bc}\delta(\boldsymbol{k}-\boldsymbol{k}')\,. \end{aligned}\right. \tag{6.14}$$

The vacuum is defined as $a_b(\boldsymbol{k})\,|\Psi_0\rangle = 0$. From (6.12) and (6.13) one gets

$$\left\{\begin{aligned} &[A_b(x),\, A_c(x')]_- && = -i\eta_{bc}D(xx')\,, \\ &[A_b(\boldsymbol{x}, x^4),\, A_c(\boldsymbol{x}', x^4)]_- && = 0\,, \\ &[\pi_b(\boldsymbol{x}, 0),\, A_c(\boldsymbol{x}', 0)]_- && = [\Delta'_4 A_b(\boldsymbol{x}, 0),\, A_c(\boldsymbol{x}', 0)] = i\eta_{bc}\delta_{\mathbf{x}\mathbf{x}'}\,, \\ &[A_b^{(-)}(x),\, A_c^{(-)}(x')]_- && = [A_b^{(+)}(x),\, A_c^{(+)}(x')]_- = 0\,, \\ &[A_b^{(-)}(x),\, A_c^{(+)}(x')]_- && = -i\eta_{bc}D_+(xx')\,, \\ &[A_b^{(+)}(x),\, A_c^{(-)}(x')]_- && = -i\eta_{bc}D_-(xx')\,, \\ &[F_{ab}(x),\, F_{cd}(x')]_- && = i[\eta_{a\cdot}\Delta'_b\Delta'_{d'} + \eta_{b\cdot}\Delta'_a\Delta'_{c'} - \eta_{bc}\Delta'_a\Delta'_{d'} - \eta_{ad}\Delta'_b\Delta'_{c'}]D(xx')\,, \end{aligned}\right. \tag{6.15}$$

where $F_{ab}(x) = -F_{ba}(x) = \Delta'_b A_a(x) - \Delta'_a A_b(x)$.

The energy and momentum of the photon field are given by

$$(6.16)\qquad \begin{cases} \mathrm{H}(0) = \int_{-\pi}^{\pi}\int\int\int \sum_{(a)} \mathrm{d}^3\boldsymbol{k}\,[N_{(a)}(\boldsymbol{k}) + 2]\,\underset{(0)}{\omega}(\boldsymbol{k})\sqrt{1 - \underset{(0)}{\omega}{}^2/4} \\ P_{\mathbf{a}}(0) = \int_{-\pi}^{\pi}\int\int\int \sum_{(a)} \mathrm{d}^3\boldsymbol{k} N_{(a)}(\boldsymbol{k})\,[\sin k_{\mathbf{a}}]\,, \end{cases}$$

where $N_{(b)}(\boldsymbol{k}) = -a^*{}_b(\boldsymbol{k})a^b(\boldsymbol{k})$ [$N_{(4)}$ being negative], and the energy is not positive definite, as we have not exploited the subsidiary condition.

iii) Free electron-positron field: The Lagrangian and the field equations are respectively

$$(6.17)\qquad \begin{cases} L = -\tfrac{1}{2}\overline{\psi}(x)[-i\gamma^a\Delta'_a + mI]\psi(x) - \tfrac{1}{2}\psi(x)[i\gamma^a\Delta'_a + mI]\psi(x)\,, \\ [-i\gamma^a\Delta_a + mI]\psi(x) = 0\,, \\ \overline{\psi}(x)[i\gamma^a\overline{\Delta}_a + mI] = 0\,, \\ \overline{\psi}(x) = \psi^*(x)\gamma^4\,. \end{cases}$$

A typical plane-wave solution is characterized by $\psi(x) \quad u(p)\exp[-ip_a x^a]$. From (6.17) one easily obtains

$$(6.18)\qquad \sin p_4 = \pm\sqrt{\sin^2 p_1 + \sin^2 p_2 + \sin^2 p_3 + m^2} = \pm|E(\boldsymbol{p})|.$$

There are four independent plane-wave solutions $u^{(r)}(\boldsymbol{p})$, $v^{(r)}(\boldsymbol{p})$ $(r = 1, 2)$ of the Dirac equations. The normalization adopted is

$$(6.19)\qquad \begin{cases} \overline{u}{}^{(r)}_{\alpha}(\boldsymbol{p})u^{(s)}_{\alpha}(\boldsymbol{p}) = \delta^{(rs)}, \\ u^{*(r)}_{\alpha}(\boldsymbol{p})u^{(s)}_{\alpha}(\boldsymbol{p}) = \dfrac{|E(\boldsymbol{p})|}{m}\,\delta^{(rs)}, \\ \overline{v}{}^{(r)}_{\alpha}(\boldsymbol{p})v^{(s)}_{\alpha}(\boldsymbol{p}) = -\delta^{(rs)}, \\ v^{*(r)}_{\alpha}(\boldsymbol{p})v^{(s)}_{\alpha}(\boldsymbol{p}) = \dfrac{|E(\boldsymbol{p})|}{m}\,\delta^{(rs)}, \\ \overline{u}{}^{(r)}_{\alpha}(\boldsymbol{p})v^{(s)}_{\alpha}(\boldsymbol{p}') = \overline{v}{}^{(r)}_{\alpha}(\boldsymbol{p})u^{(s)}_{\alpha}(\boldsymbol{p}') = 0\,, \\ \sum_{(r)=1}^{2} u^{(r)}_{\alpha}(\boldsymbol{p})\,\overline{u}{}^{(r)}_{\beta}(\boldsymbol{p}) = \Lambda_{+\alpha\beta}(\boldsymbol{p})\,, \\ \sum_{(r)=1}^{2} v^{(r)}_{\alpha}(\boldsymbol{p})\,\overline{v}{}^{(r)}_{\beta}(\boldsymbol{p}) = -\Lambda_{-\alpha\beta}(\boldsymbol{p})\,, \end{cases}$$

where the projection operators are defined as

$$\Lambda_{\pm}(\boldsymbol{p}) = \frac{m \pm \gamma^a \sin p_a}{2m}, \qquad \Lambda_+ + \Lambda_- = 1, \qquad \Lambda_{\pm}^2 = \Lambda_{\pm}, \qquad \Lambda_{\pm}\Lambda_{\mp} = 0 . \tag{6.20}$$

The truncated Fourier transform of the spinor field leads to

$$\begin{cases} \psi(x) = \dfrac{1}{(2\pi)^{\frac{3}{2}}} \sum_{r=1}^{2} \int_{-\pi}^{\pi}\int\int\int \sqrt{\dfrac{m}{|E(\boldsymbol{p})|\sqrt{[1-E^2(\boldsymbol{p})]}}}\, \mathrm{d}^3\boldsymbol{p}\, [b_{(r)}(\boldsymbol{p})\, u^{(r)}(\boldsymbol{p}) \cdot \\ \qquad \cdot \exp[-i\boldsymbol{p}_a x^a] + d^*_{(r)}(\boldsymbol{p})\, v^{(r)}(\boldsymbol{p}) \exp[i\boldsymbol{p}_a x^a]] = \psi^{(-)}(x) + \psi^{(+)}(x) , \\ \bar{\psi}(x) = \dfrac{1}{(2\pi)^{\frac{3}{2}}} \sum_{r=1}^{2} \int_{-\pi}^{\pi}\int\int\int \sqrt{\dfrac{m}{|E(\boldsymbol{p})|\sqrt{[1-E^2(\boldsymbol{p})]}}}\, \mathrm{d}^3\boldsymbol{p}\, [b^*_{(r)}(\boldsymbol{p})\, \bar{u}^{(r)} \exp[i\boldsymbol{p}_a x^a] + \\ \qquad + d_{(r)}(\boldsymbol{p})\, \bar{v}^{(r)}(\boldsymbol{p}) \exp[-i\boldsymbol{p}_a x^a]] = \bar{\psi}^{(+)}(x) + \bar{\psi}^{(-)}(x) . \end{cases} \tag{6.21}$$

The quantum rules are

$$\begin{cases} [b_{(r)}(\boldsymbol{p}), b_{(s)}(\boldsymbol{p}')]_+ = [b^*_{(r)}(\boldsymbol{p}), b^*_{(s)}(\boldsymbol{p}')]_+ = [d_{(r)}(\boldsymbol{p}), d_{(s)}(\boldsymbol{p}')]_+ = \\ \qquad\qquad = [d^*_{(r)}(\boldsymbol{p}'), d^*_{(s)}(\boldsymbol{p}')]_+ = 0 , \\ [b_{(r)}(\boldsymbol{p}), d_{(s)}(\boldsymbol{p}')]_+ = [b^*_{(r)}(\boldsymbol{p}), d^*_{(s)}(\boldsymbol{p}')]_+ = \\ \qquad\qquad = [b_{(r)}(\boldsymbol{p}), d^*_{(s)}(\boldsymbol{p}')]_+ = [b^*_{(r)}(\boldsymbol{p}), d_{(s)}(\boldsymbol{p}')]_+ = 0 , \\ [b_{(r)}(\boldsymbol{p}), b^*_{(s)}(\boldsymbol{p}')]_+ = \delta_{(rs)}\delta(\boldsymbol{p}-\boldsymbol{p}') , \\ [d^*_{(r)}(\boldsymbol{p}), d_{(s)}(\boldsymbol{p}')]_+ = \delta_{(rs)}\delta(\boldsymbol{p}-\boldsymbol{p}') . \end{cases} \tag{6.22}$$

The vacuum is characterized by $b_{(r)}(\boldsymbol{p})\,|\Psi_0\rangle = d_{(r)}(\boldsymbol{p})\,|\Psi_0\rangle = 0$. From (6.21) and (6.22) one arrives at

$$\begin{cases} [\psi_\alpha(x), \bar{\psi}_\beta(x')]_+ = -iS_{\alpha\beta}(xx') , \\ [\psi_\alpha(\boldsymbol{x}, 0), \psi^*_\beta(\boldsymbol{x}', 0)]_+ = \delta_{\alpha\beta}\delta_{\boldsymbol{x}\boldsymbol{x}'} , \\ [\psi^{(-)}_\alpha(x), \bar{\psi}^{(-)}_\beta(x')]_+ = [\psi^{(+)}_\alpha(x), \bar{\psi}^{(+)}_\beta(x')]_+ = 0 , \\ [\psi^{(-)}_\alpha(x), \psi^{(+)}_\beta(x)]_+ = [\bar{\psi}^{(+)}_\alpha(x), \bar{\psi}^{(-)}_\beta(x)]_+ = 0 , \\ [\psi^{(-)}_\alpha(x), \bar{\psi}^{(+)}_\beta(x')]_+ = -iS_{+\alpha\beta}(xx') , \\ [\psi^{(+)}_\alpha(x), \bar{\psi}^{(-)}_\beta(x')]_+ = -iS_{-\alpha\beta}(xx') . \end{cases} \tag{6.23}$$

The energy, momentum and charge of the electron-positron field are given by

$$
(6.24)\qquad
\begin{cases}
\mathrm{H}(0) = \sum_{r=1}^{2} \int\int\int_{-\pi}^{\pi} \mathrm{d}^3\boldsymbol{p}\ |E(\boldsymbol{p})| \left[N_{-(r)}(\boldsymbol{p}) + N_{+(r)}(\boldsymbol{p}) - 1 \right], \\
P_{\mathbf{a}}(0) = \sum_{r=1}^{2} \int\int\int_{-\pi}^{\pi} \mathrm{d}^3\boldsymbol{p} \sin \boldsymbol{p}_{\mathbf{a}} \left[N_{-(r)}(\boldsymbol{p}) + N_{+(r)}(\boldsymbol{p}) \right], \\
Q(0) = e \sum_{r=1}^{2} \int\int\int_{-\pi}^{\pi} \mathrm{d}^3\boldsymbol{p} \left[- N_{-(r)}(\boldsymbol{p}) + N_{+(r)}(\boldsymbol{p}) - 1 \right], \\
|\underset{\text{vac}}{\mathrm{H}}(0)| < (2\pi)^3 \sqrt{3 + m^2}, \qquad \underset{\text{vac}}{Q}(0) = -(2\pi)^3 e,
\end{cases}
$$

where electron and positron number operators are

$$
b^*_{(r)}(\boldsymbol{p}) b_{(r)}(\boldsymbol{p}) = N_{-(r)}(\boldsymbol{p}), \qquad d^*_{(r)}(\boldsymbol{p}) d_{(r)}(\boldsymbol{p}) = N_{+(r)}(\boldsymbol{p}), \quad \text{respectively.}
$$

7. – Interacting fields and the S-matrix.

In the interaction picture, the field operators satisfy the free field equations and the corresponding commutation relations, whereas the discrete evolution of the state vector of the interacting system is governed by

$$
(7.1)\qquad i \Delta'_4 |\Psi(x^4)\rangle = \underset{\text{int}}{\mathrm{H}}(x^4) |\Psi(x^4)\rangle,
$$

where ([47])

$$
\underset{\text{int}}{\mathrm{H}}(x^4) = \sum_{\text{space-cells}} \underset{\text{int}}{\mathscr{H}}(\boldsymbol{x}, x^4).
$$

We shall introduce the unitary operator U through ([48])

$$
(7.2)\qquad
\begin{cases}
|\Psi(\underset{(2)}{x^4})\rangle = U(\underset{(2)}{x^4}, \underset{(1)}{x^4}) |\Psi(\underset{(1)}{x^4})\rangle, \\
U(\underset{(2)}{x^4}, \underset{(1)}{x^4}) = 1 - i \sum_{x^4 = \underset{(1)}{x^4} + 1}^{\underset{(2)}{x^4}} \underset{\text{int}}{\mathrm{H}}(x^4)\, U(x^4, \underset{(1)}{x^4}).
\end{cases}
$$

([47]) It may be mentioned here that the difficulty of the non-local interactions with arbitrary form factors in satisfying the so-called integrability condition does not trouble us. The intercellular causality condition $[\underset{\text{int}}{\mathscr{H}}(\boldsymbol{x}, x^4), \underset{\text{int}}{\mathscr{H}}(\boldsymbol{x}', x^4)]_- = 0$ is satisfied, and intracellular causality has no meaning in our point of view.

The S-matrix is defined by

$$S = \lim_{\substack{x^4_{(2)} \to \infty \\ x^4_{(1)} \to -\infty}} U(x^4_{(2)}, x^4_{(1)}) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \sum_{x_{(1)}=-\infty}^{\infty} \cdots \sum_{x_{(n)}=-\infty}^{\infty} P[\mathscr{H}_{\text{int}}(x_{(1)}) \cdots \mathscr{H}_{\text{int}}(x_{(n)})], \tag{7.3}$$

where P is Dyson's chronological operator. In the Newmann-Liouville expansion of the S-matrix (given by (7.3)) at least the semiconvergence of the series is assumed.

For the reduction of the S-matrix we shall introduce the normal product operator N and Wick's chronological operator $T = \delta_p P$. The contraction of the two field operators is defined by $(T - N) A(x_{(1)}) B(x_{(2)}) = A(x_{(1)}) B(x_{(2)})$. Wick's theorem of decomposition states that

$$\begin{aligned} T\,N\{A(x_{(1)})B(x_{(1)})C(x_{(1)})\}\,N\{A(x_{(2)})B(x_{(2)})C(x_{(2)})\}\ldots] = \\ = [N\{A(x_{(1)})B(x_{(1)})C(x_{(1)})\}\,N\{A(x_{(2)})B(x_{(2)})C(x_{(2)})\}\ldots] + \\ + [N\{A(x_{(1)})B(x_{(1)})\,C(x_{(1)})\}\,N\{A(x_{(2)})B(x_{(2)})C(x_{(2)})\}\ldots + \ldots] + \\ + [N\{A(x_{(1)})B(x_{(1)})C(x_{(1)})\}\,N\{A(x_{(2)})B(x_{(2)})C(x_{(2)})\}\ldots + \ldots] + \ldots \end{aligned} \tag{7.4}$$

From the commutation relations given in Section 6, one can verify

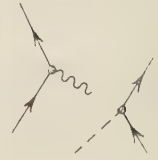
$$\begin{cases} \overline{\psi}_\alpha(x)\,\overline{\psi}_\beta(x') = \psi_\alpha(x)\psi_\beta(x') = \overline{\psi}_\alpha(x)\varphi(x') = \psi_\alpha(x)\,\varphi(x') = \\ \qquad\qquad = \overline{\psi}_\alpha(x)\,A_b(x') = \psi_\alpha(x)\,A_b(x') = 0\,, \\ \varphi(x)\varphi(x') = -i\,\Delta_F(xx')\,, \qquad A_b(x)A_c(x') = i\eta_{bc}\,D_F(xx')\,, \\ \overline{\psi}_\alpha(x)\,\psi_\beta(x') = -iS_{F^\alpha\beta}(xx')\,, \qquad \psi_\alpha(x)\,\overline{\psi}_\beta(x') = iS_{F^\alpha\beta}(xx')\,. \end{cases} \tag{7.5}$$

The interaction energy-densities for the electrodynamics and meson-nucleon processes are respectively

$$\begin{cases} \mathscr{H}_{\text{int}}(x) = e\,N[\bar{\psi}_\alpha(x)(\gamma^a)_{\alpha\beta}A_a(x)\psi_\beta(x)], \\ \mathscr{H}_{\text{int}}(x) = g\,N[\bar{\psi}_\alpha(x)(\Gamma)_{\alpha\beta}\varphi(x)\psi_\beta(x)]\,, \end{cases} \tag{7.6}$$

(48) For Heisenberg representation the unitary transformation through $U(x_4)$ leads to complication.

TABLE I. – *Feynman-Dyson prescriptions for cellular space-time model.*

Factor in S-matrix (in position space)	Factor in S-matrix (in momentum space)	Approximate rules	Component of diagram
Meson propagator: $\varphi(x)\varphi(x') = -i\,\Delta_F(xx')$.	$\lim_{\varepsilon\to 0} \dfrac{-i}{(2\pi)^4 - 4\eta^{ab}\sin k_a/2 \sin k_b/2 + \mu^2 - i\varepsilon}$	$\dfrac{i}{(2\pi)^4}\,\dfrac{1}{-k_a k^a + \mu^2}$	
Photon propagator: $A_a(x)\,A_b(x') = i\eta_{ab}\,D_F(xx')$	$\lim_{\varepsilon\to 0} \dfrac{i}{(2\pi)^4}\,\dfrac{\eta_{ab}}{-4\eta^{cd}\sin k_c/2 \sin k_d/2 - i\varepsilon}$	$\dfrac{i}{(2\pi)^4}\,\dfrac{\eta_{ab}}{-k_c k^c}$	
Electron-positron (or nucleon-antinucleon) propagators: $-\overline{\psi}_\alpha(x)\psi_\beta(x') = \psi_\alpha(x)\overline{\psi}_\beta(x') = iS_{F\alpha\beta}(xx')$	$\lim_{\varepsilon\to 0} \dfrac{-i}{(2\pi)^4}\,\dfrac{(\gamma^a \sin p_a + mI)_{\alpha\beta}}{-\eta^{bc}\sin p_b \sin p_c + m^2 - i\varepsilon}$	$\dfrac{-i}{(2\pi)^4}\,\dfrac{(\gamma^a p_a + mI)_{\alpha\beta}}{-p_b p^b + m^2}$	
Electron-positron photon corner: $e(\gamma^a)_{\alpha\beta}$	$(2\pi)^4 e(\gamma^a)_{\alpha\beta}\,\delta^4(p - p' \pm k)$	$(2\pi)^4\, e(\gamma^a)_{\alpha\beta}\,\delta^4(p - p' \pm k)$	
Nucleon-antinucleon meson corner: $g(\Gamma)_{\alpha\beta}$	$(2\pi)^4 g(\Gamma)_{\alpha\beta}\,\delta^4(p - p' \pm k)$	$(2\pi)^4\, g(\Gamma)_{\alpha\beta}\,\delta^4(p - p' \pm k)$	
Incoming or outgoing external meson lines:	$\dfrac{1}{(2\pi)^{\frac{3}{2}}}\,\dfrac{1}{\sqrt{2\,\lvert\omega(\boldsymbol{k})\rvert}\,\sqrt{1 - \omega^2(\boldsymbol{k})/4}}$,	$\dfrac{1}{(2\pi)^{\frac{3}{2}}}\,\dfrac{1}{\sqrt{2\,\lvert\overline{\omega}(\boldsymbol{k})\rvert}\,\sqrt{1 - \overline{\omega}^2(\boldsymbol{k})/4}}$,	

Incoming or outgoing external photon lines: $A_b^{(-)}(x)$, $A_b^{(+)}(x)$	$(2\pi)^{\frac{3}{2}} \sqrt{2}\, \lvert \underset{(1)}{\omega}(\boldsymbol{k}) \rvert \sqrt{1 - \underset{(0)}{\omega^2}/4}$ $\lvert \underset{(0)}{\omega}(\boldsymbol{k}) \rvert = \lim_{\mu \to 0} \lvert \omega(\boldsymbol{k}) \rvert$	$(2\pi)^{\frac{3}{2}} \sqrt{2}\, \lvert \underset{(0)}{\overline{\omega}}(\boldsymbol{k}) \rvert \sqrt{1 - \underset{(0)}{\overline{\omega}^2}(\boldsymbol{k})/4}$ $\lvert \underset{(0)}{\overline{\omega}}(\boldsymbol{k}) \rvert = \lim_{\mu \to 0} \lvert \overline{\omega}(\boldsymbol{k}) \rvert$	
Incoming or outgoing external electron-positron (or nucleon-antinucleon) lines: $\psi_\lambda^{(-)}(x)$, $\overline{\psi}_\lambda^{(+)}(x)$; $\overline{\psi}_\lambda^{(-)}(x)$, $\psi_\lambda^{(+)}(x)$	$\frac{1}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{m}{\lvert (\boldsymbol{p}) \rvert \sqrt{1 - E^2(\boldsymbol{p})}}} \cdot$ $\cdot \left[u_\lambda^{(r)}(\boldsymbol{p}), u_\lambda^{(r)}(\boldsymbol{p}); v_\lambda^{(r)}(\boldsymbol{p}), v_\lambda^{(r)}(\boldsymbol{p}) \right]$, $\lvert E(\boldsymbol{p}) \rvert = \sqrt{\sin^2 p_1 + \sin^2 p_2 + \sin^2 p_3 + m^2}$	$\frac{1}{(2\pi)^{\frac{3}{2}}} \sqrt{\frac{m}{\lvert \bar{E}(\boldsymbol{p}) \rvert \sqrt{1 - \bar{E}^2(\boldsymbol{p})}}} \cdot$ $\cdot \left[u_\lambda^{(r)}(\boldsymbol{p}), \overline{u}_\lambda^{(r)}(\boldsymbol{p}); \overline{v}_\lambda^{(r)}(\boldsymbol{p}), v_\lambda^{(r)}(\boldsymbol{p}) \right]$, $\lvert \bar{E}(\boldsymbol{p}) \rvert = \sqrt{p_1^2 + p_2^2 + p_3^2 + m^2}$	
Sum over all space-time cells placing a factor $(-i)^n/n!$ in front of a n-th order term	Integrate over virtual or internal 4-« momenta » from $-\pi$ to π: $\int_{-\pi}^{\pi}\int\int\int [\ldots] \mathrm{d}^4 k$.	$\int_{-\pi}^{\pi}\int\int\int [\ldots] \mathrm{d}^4 k =$ $= 2\pi^2 i \int_0^{2\pi > M > \pi} [\ldots] \lvert k \rvert^3 \mathrm{d} \lvert k \rvert$	

where $(\Gamma)_{\lambda\beta}$ is $\delta_{\lambda\beta}$ or $(\gamma_5)_{\lambda\beta}$ according to scalar or pseudoscalar meson-field. Therefore, the S-matrices for these two systems are respectively (from (7.3) and (7.6)),

$$(7.7)\quad \begin{cases} S = \sum_{n=0}^{\infty} \frac{(-ie)^n}{n!} \sum_{\substack{x=-\infty \\ (1)}}^{\infty} \dots \sum_{\substack{x=-\infty \\ (n)}}^{\infty} T\big[N\{\overline{\psi}_\lambda(\underset{(1)}{x})(\gamma^a)_{\lambda\beta} A_a(\underset{(1)}{x})\psi_\beta(\underset{(1)}{x})\} \dots \cdot \\ \qquad \cdot N\{\overline{\psi}_\alpha(\underset{(n)}{x})(\gamma^a)_{\alpha\beta} A_a(\underset{(n)}{x})\psi_\beta(\underset{(n)}{x})\}\big], \\ S = \sum_{n=0}^{\infty} \frac{(-ig)^n}{n!} \sum_{\substack{x=-\infty \\ (1)}}^{\infty} \dots \sum_{\substack{x=-\infty \\ (n)}}^{\infty} T\big[N\{\overline{\psi}_\alpha(\underset{(1)}{x})(\Gamma)_{\alpha\lambda}\varphi(\underset{(1)}{x})\psi_\beta(\underset{(1)}{x})\} \dots \cdot \\ \qquad \cdot N\{\overline{\psi}_\alpha(\underset{(n)}{x})(\Gamma)_{\lambda\beta}\varphi(\underset{(n)}{x})\psi_\beta(\underset{(n)}{x})\}\big]. \end{cases}$$

For the practical evaluation of (7.7) one has to decompose a typical n-th order term by (7.4) and (7.5). Then one has to transform all the field operators and causal functions into « momentum » space, and sum over all space-time cells, and integrate over all internal « moment ». The correspondence of each decomposed term with a simple Feynman-Dyson graph, and the appropriate prescriptions, are supplied in the accompanying Table I.

8. – Question of semi-convergence of the S-matrix series.

In this section, we shall prove the finiteness of any typical term of the S-matrix series for meson-nucleon processes (which is free from infrared ambiguities). But it is difficult to say anything rigorously about the structure of the S-matrix series, because the exact evaluation of a typical higher order term of the S-matrix series is well nigh impossible, and even the lower bound is difficult to find. So we shall be satisfied to work with the upper bound of a typical term. We shall call the resulting series (with upper bounds of each term) the « gross » S-matrix series, and prove its Borel summability. Though the « gross » series is different from the true series, still it might well be conjectured that the behaviour of the true series would not be radically different from that of the « gross » series. Throughout this section it will be assumed that the number of external lines is not too large, and the frequencies or wave-numbers associated with them are small compared to π.

An element of the S-matrix for a typical transition can be written as $\langle\Psi_b|S|\Psi_a\rangle$. Let the total number of the external meson and nucleon-anti-nucleon lines in this process be E_m $(= E_m^{(a)} + E_m^{(b)})$ and E_N $(= E_N^{(a)} + E_N^{(b)}$, which is even), respectively. Then $\langle\Psi_b|S|\Psi_a\rangle$ can be split up into ([49,50])

$$(8.1)\quad \langle\Psi_b|S|\Psi_a\rangle = \langle\Psi_0| \underset{(E_N+E_m)}{S} |\Psi_0\rangle + \sum_i \sum_{i'} \delta^4(l_{(i)} - k_{(i')})\langle\Psi_0| \underset{(E_N+E_m-2)}{S} |\Psi_0\rangle + \dots,$$

([49]) D. Judge, H. Shimodaira and Y. Takahashi: *Nuovo Cimento* **16**, 1139 (1960).
([50]) D. Judge, H. Shimodaira and Y. Takahashi: *Proc. Roy. Irish Acad.*, Sect. A (to be published).

where

$$S_{(E_N+E_m)} = (-1)^m \Big[[\dots [[[\dots [[\dots [S, a^+(\boldsymbol{k}_{(1)})]_- \dots \cdot$$
$$\cdot a^+(\boldsymbol{k}_{(E_m^{(a)})})]_- a(\boldsymbol{k}'_{(1)})]_- \dots a(\boldsymbol{k}'_{(E_m^{(b)})})]_- c^+(\boldsymbol{p}_{(1)})]_- c^+(\boldsymbol{p}_{(2)})]_+ \dots]_- c(\boldsymbol{p}'_{(E_N^{(b)})})\Big]_+ ,$$

where $c(\boldsymbol{p})$ stands for either $b(\boldsymbol{p})$ or $d(\boldsymbol{p})$.

In this expansion, the first term corresponds to the process where all external lines participate in interactions, and the second term corresponds to the process where all but one participate, and so on. We shall restrict ourselves to the first term because the following arguments can be carried over to the others. We can write

$$(8.2) \qquad \begin{cases} \langle \Psi_0 | S_{(E_m+E_N)} | \Psi_0 \rangle = \delta^4 \left(\sum p' - \sum p \right) R_{ba} , \\ R_{ba} = \sum_{n=E_m+E_N-2}^{\infty} \dfrac{(-ig)^n}{n!} N(n, E_m, E_N) M_{ba}^{(n)} , \end{cases}$$

where $M_{ba}^{(n)}$ is a typical matrix element corresponding to the graph with n corners and external lines E_m, E_N; $N(n, E_m, E_N)$ is the number of possible graphs with the same specifications. It can be easily seen that the R-series is either in odd or even powers of g according as E_m is odd or even. We shall consider an even series

$$(8.3) \qquad R_{ba} + R_0 = \sum_{n=0}^{\infty} (-1)^n \frac{(g)^{2n}}{(2n)!} N(2n, E_m, E_N) M_{ba}^{(2n)} ,$$

where we have added some initial term R_0 so that the series starts from $n = 0$.

The enumeration of graphs ([51]) is given by the monomials in the polynomial $n! (-i/\sqrt{2})^n H_n(-i\xi/\sqrt{2}) \mathscr{L}_n(-xy)$, where H_n, $\mathscr{L}_n$ are Hermite and Laguerre polynomials. The number $N(2n, E_m, E_N)$ can be expressed as

$$(8.4) \qquad \begin{cases} N(2n, E_m, E_N) = (2n)!^3 / \{(2n - \tfrac{1}{2}E_N)! [\tfrac{1}{2}(2n - E_m)]! E_m! [(E_N/2)!]^2 2^{\frac{1}{2}(2n - E_m)}\} \simeq \\ \qquad \simeq [(2n)!]^2 / \{n! 2^n E_m! [(E_N/2)!]^2 2^{-E_m/2}\} . \end{cases}$$

Now, to calculate the upper bound of $M_{ba}^{(2n)}$ we notice the following relations

$$(8.5) \qquad \begin{cases} Q = \tfrac{1}{2}(2n - E_m - E_N) + 1 , \\ I_N = 2n - \tfrac{1}{2}E_N , \\ I_m = \tfrac{1}{2}(2n - E_m) , \end{cases}$$

([51]) C. A. Hurst: *Proc. Roy. Soc.*, A **214**, 44 (1952); *Proc. Camb. Phil. Soc.*, **48**, 625 (1952).

where ϱ, I_N, I_m are the number of independent integrations over internal «momenta», internal nucleon-antinucleon lines and internal meson lines.

For the integration over «internal momenta» one can rotate the path C_F in the complex k_4-plane by $\pi/2$ without crossing poles and neglecting the contribution from arcs, so that the upper bounds of the contributions from internal meson and nucleon lines can be written as

$$(8.6)\quad \begin{cases} \dfrac{1}{(2\pi)^4}\left|\dfrac{1}{4(\sin^2 k_1/2+\sin^2 k_2/2+\sin^2 k_3/2+\sinh^2 k_4/2)+\mu^2}\right| < \dfrac{1}{(2\pi)^4}\dfrac{1}{\mu^2}, \\ \dfrac{1}{(2\pi)^4}\left|\dfrac{\gamma^1\sin p_1+\gamma^2\sin p_2+\gamma^3\sin p_3+i\gamma^4\sinh p_4+mI}{4(\sin^2 p_1+\sin^2 p_2+\sin^2 p_3+\sinh^2 p_4)+m^2}\right| < \\ \qquad < \dfrac{1}{(2\pi)^4}\left|\dfrac{3+\sinh\pi+m}{m^2}\right| I. \end{cases}$$

From (8.5) and (8.6) we can obtain the upper bound of $M_{ba}^{(2n)}$ as

$$(8.7)\quad |M_{ba}^{(2n)}| < \bar{M}_{ba}^{(2n)} = (2\pi)^{8n}\frac{1}{(2\pi)^{4(I_N+I_m)}}\cdot$$

$$\cdot\left|\frac{m^{E_N/2}}{2^{E_m/2}(2\pi)^{\frac{3}{2}(E_N+E_m)}}\prod_{r=1}^{E_m}\prod_{s=1}^{E_N}\frac{1}{\sqrt{|\omega(\boldsymbol{k}_{(r)})|}\sqrt{[1-\omega^2(\boldsymbol{k}_{(r)})/4]}}\cdot\right.$$

$$\left.\cdot\frac{1}{\sqrt{|E(\boldsymbol{p}_{(s)})|}\sqrt{[1-E^2(\boldsymbol{p}_{(s)})]}}\right|\cdot\int_{-\pi}^{\pi}\ldots\int_{-\pi}^{\pi}\mathrm{d}^4k_{(1)}\mathrm{d}^4k_{(2)}\ldots\mathrm{d}^4k_{(\varrho)}\left\{\frac{1}{\mu^2}\right\}^{I_m}\left\{\frac{3+\sinh\pi+m}{m^2}\right\}^{I_N} =$$

$$= k(E)\left\{\frac{3+\sinh\pi+m}{m^2\mu}\right\}^{2n},$$

where

$$k(E) = (2\pi)^4\left\{\frac{\mu^2}{2(2\pi)^3}\right\}^{E_m/2}\left\{\frac{m^3}{(2\pi)^3(3+\sinh\pi+m)}\right\}^{E_N/2}\cdot$$

$$\cdot\prod_{r=1}^{E_m}\prod_{s=1}^{E_N}\frac{1}{\sqrt{|\omega(\boldsymbol{k}_{(r)})|}\sqrt{[1-\omega^2(\boldsymbol{k}_{(r)})/4]}}\frac{1}{\sqrt{|E(\boldsymbol{p}_{(s)})|}\sqrt{[1-E^2(\boldsymbol{p}_{(s)})]}}.$$

From (8.7) it follows that $|M_{bc}^{(2n)}| < \infty$ for any finite n and small external «momenta». This is an important result in this investigation.

Now, instead of the true series (8.3) we shall investigate the properties of the «gross» series defined by

$$(8.8)\qquad \bar{R}_{ab}+\bar{R}_0 = \sum_{n=0}^{\infty}(-1)^n\frac{g^{2n}}{(2n)!}N(2n, E_m, E_N)\bar{M}_{ba}^{(2n)}.$$

This is an alternating asymptotic ([52-54]) (or semi-convergent) series, its « Borel sum » being (from (8.4), (8.7))

$$(8.9)\qquad \bar{R}_{ab}+\bar{R}_0 = k'(E)\int_0^\infty \exp\left[-t-(tg')^2/2\right]\mathrm{d}t = \\ = k'(E)\sqrt{(\pi/2)}(g')^{-1}\exp\left[1/(2g')^2\right]\left\{1-\Phi[1/\sqrt{2}g']\right\},$$

where

$$k'(E)=\frac{k(E)}{2^{-E_m/2}E_m!\left[(E_N/2)!\right]^2}, \qquad g'=g\left\{\frac{3+\sinh\pi+m}{\mu m^2}\right\},$$

$$\Phi(x)=\frac{2}{\sqrt{\pi}}\int_0^x \exp\left[-t^2\right]\mathrm{d}t\,.$$

From (8.9) it is evident that the « Borel sum » exists for all real values of the coupling constant g and is analytic for $\mathrm{Re}\,g^2>0$, but cannot be continued along the imaginary axis of g.

9. – A rough estimate of the fundamental length.

Using the perturbation prescription for cellular space and continuous time model one can calculate the potential of two fixed nucleons or electrons due to one instantaneous exchange of meson or photon. The results are

$$(9.1)\qquad \begin{cases} V[\boldsymbol{x}_{(n)}-\boldsymbol{x}_{(n')}] = \dfrac{g^2}{(2\pi)^3}\,\Gamma_{(n)}\,\Gamma_{(n')}\displaystyle\int_{-\pi}^{\pi}\!\!\int\!\!\int \frac{\exp\left[-ik_{\mathbf{a}}[x^{\mathbf{a}}_{(n)}-x^{\mathbf{a}}_{(n')}]\right]\mathrm{d}^3\boldsymbol{k}}{4(\sin^2 k_1/2+\sin^2 k_2/2+\sin^2 k_3/2)+\mu^2}, \\ V[\boldsymbol{x}_{(n)}-\boldsymbol{x}_{(n')}] = -\dfrac{e^2}{(2\pi)^3}\displaystyle\sum_{(\lambda)}\int_{-\pi}^{\pi}\!\!\int\!\!\int \cdot \\ \qquad\qquad \cdot\frac{\left[\gamma^a_{(n)}\varepsilon_a^{(\lambda)}(\boldsymbol{k})\right]\left[\gamma^b_{(n')}\varepsilon_b^{(\lambda)}(\boldsymbol{k})\right]\exp\left[-ik_{\mathbf{a}}\{x^{\mathbf{a}}_{(n)}-x^{\mathbf{a}}_{(n')}\}\right]\mathrm{d}^3\boldsymbol{k}}{4(\sin^2 k_1/2+\sin^2 k_2/2+\sin^2 k_3/2)}. \end{cases}$$

([52]) F. J. Dyson: *Phys. Rev.*, **85**, 631 (1952).
([53]) S. Hori: *Progr. Theor. Phys.*, **8**, 569 (1952).
([54]) R. J. Riddell: *Phys. Rev.*, **91**, 1243 (1953).

The self-energy of the electron becomes ([55])

$$(9.2) \qquad V(0) = \frac{2e^2}{(2\pi)^3} \int_{-\pi}^{\pi}\int\int \frac{\mathrm{d}^3\boldsymbol{k}}{4(\sin^2 k_1/2 + \sin^2 k_2/2 + \sin^2 k_3/2)} =$$

$$= e^2[18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6}]\left[\frac{2}{\pi} K\{(2-\sqrt{3})(\sqrt{3}-\sqrt{2})\}\right]^2,$$

where K denotes the complete elliptic integral.

It may be mentioned that even from the Feynman-Dyson prescription for the cellular space-time model the electron self-energy does not vanish in the absence of bare mass. Equating (9.2) with the electron mass one gets an estimate of the fundamental length as $l \sim 1.7 \cdot 10^{-12}$ cm. But this value is not reliable, as it is calculated on the basis of old time perturbation theory. Noticing the similarity between our perturbation prescriptions (which are not explicitly given in this paper) and Chew's, if we use his cut-off value of 6 meson masses, then the estimate of the fundamental length becomes $l \sim 2 \cdot 10^{-14}$ cm. For this value of l, the upper limits for fermion and boson masses will be of the order of one and two nucleon masses respectively.

* * *

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([55]) G. N. Watson: *Quart. Journ. Math.*, **10**, 266 (1939).

RIASSUNTO (*)

Si introduce un semplice modello cellulare dello spazio-tempo, comprendente in modo naturale una lunghezza fondamentale. Il concetto usuale delle quantità di campo come funzioni dei punti geometrici viene respinto a favore del concetto delle quantità di campo come funzionali delle celle elementari. Conseguentemente le usuali equazioni di campo sotto forma di equazioni differenziali parziali sono sostituite dalle corrispondenti equazioni alle differenze parziali. La successiva quantizzazione fornisce una matrice S che non è inficiata dalla presenza di divergenze notevoli. Si dimostra che la struttura fondamentale dello spazio-tempo e la matrice S (che non comporta accoppiamenti derivati) sono covarianti per trasformazioni « discrete » di Lorentz, si determina la semi-convergenza o il comportamento asintotico della serie della matrice S, e si presenta una significativa somma di Borel della « maggiorante » della serie della matrice S.

(*) *Traduzione a cura della Redazione.*

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Regularization and Renormalization.

III – Finite-Part Integrals (*).

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Summary. — The replacement of ordinary with finite-part integrals, which generalize the definition originally given by J. Hadamard, was shown previously (1) to yield *ipso facto* correctly renormalized field equations independently of perturbative techniques, provided due care is taken of the symmetry properties of the theory and some quantitative conditions (closely related to Dyson's heuristic conditions for renormalizability) are satisfied by the theory itself. In this work we extend the class of finite-part integrals considered before to cover all instances which may practically occur and discuss several possible prescriptions in configuration space which exhibit the wanted properties. It appears that some prescriptions are better suited to the study of general questions, such as renormalizability of a theory and deduction of the Lie equations of its renormalization group, while others are more convenient for actual computation; once renormalizability is proved. This work will be followed shortly by another in which this method will be shown to yield the expected results in the case of the Lie model.

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1. – Introduction.

1'1. – In previous works (1) it was shown that a satisfactory theory of renormalization (*i.e.* both mathematically rigorous and of practical use) can be developed, provided ordinary integrals in configuration space are substituted throughout with suitably defined « finite-part » integrals $\int$, which generalize the notion first introduced by J. HADAMARD in his classical work on the Cauchy problem (2) for hyperbolic differential equations. This method is independent of perturbative techniques, since it changes directly the unrenormalized branching equations (or recursive equations among propagation kernels) into the corresponding renormalized equations, which remain therefore linear; the explicit manipulation of divergent quantities is substituted, conceptually, after a theory has been proved to be renormalizable, with the fact that the parameters in terms of which the theory is formulated are taken as indeterminate quantities, with values to be determined by comparison of that same theory with experiment. Renormalizability comes then to mean « self-consistency », that is the guarantee that the parameters initially given suffice in fact to express any number of data, or matrix elements.

This approach agrees entirely with the various attempts which have been made to substitute the usual renormalizative techniques with methods derived from the theory of distributions (3); in particular, with the concepts which are at the foundation of the masterly treatment of this subject given by N. N. BOGOLIUBOV and D. V. SHIRKOV (4). It differs from them, however, in several important respects. First, the use of appropriate combinatorial techniques permits to avoid the separate discussion of the various Feynman graphs that contribute to a given process, and thus to make full use of the remarkable symmetries which the theory is seen to possess, to the point of obtaining explicit formal expansions of all the quantities which are met. Second, it is independent of perturbative techniques and permits to obtain explicitly the differential equations of the Lie renormalization group; it permits, by the same token, to separate completely the study of the renormalizability (or consistency) of a theory, of the relations among « unrenormalized » and « renormalized » parameters, etc., from that of the questions which arise in actual cal-

(1) E. R. CAIANIELLO: *Coll. sur les Probl. Math. de la Théorie des Champs* (Lille, 1957); *Max Planck Festschr.* (1958); *Nuovo Cimento*, **13**, 637 (1959) and **14**, 185 (1959). (The latter two papers will be referred to in the text as I and II).

(2) J. HADAMARD: *Le problème de Cauchy et les equations aux dérivées partielles linéares hyperboliques* (Paris, 1932).

(3) Cfr. for instance H. J. BREMERMANN: O.N.R. Techn. Rep. no 8, Dept. of Math. Berkeley.

(4) N. N. BOGOLIUBOV and D. V. SHIRKOV: *Introduction to the Theory of Quantum Fields* (London, 1959).

culations with a theory proved, once for all, to be renormalizable. Finally, while there is no doubt that a correct use of distribution-theoretical methods would give results in agreement with ours ([5]), the fact that we can avail ourselves at all stages of the symmetry properties of the theory avoids much unnecessary toil; nor are we bound to follow in computations the explicit prescriptions of distribution theory, because we are at freedom to use a variety of methods, such as, among others, those of J. HADAMARD ([2]), M. RIESZ ([6]) J. VALATIN ([6]), or N. N. BOGOLIUBOV ([4]), in the evaluation of otherwise divergent integrals. In a few words, while from the point of view of distribution theory one is faced with the problem of defining the integral of a product of distributions as the first thing, we arrive at equivalent results by asking that the solutions of the differential branching equations belong to a well-defined mathematical class (see discussion at the end of II): the difference is more in procedure than in concept.

The present work completes that presented in I and II in several respects. While retaining all the combinatorial results reported there, it gives explicit proof that the same results remain valid also if one uses prescriptions for the evaluation of finite-part integrals more general than those assumed there; it shows at the same time that prescriptions such as those given before can be actually used, even where there has seemed to arise some doubt regarding their applicability, to the cost of very little additional combinatorial work. This is important because the explicit analysis of the equations of the renormalizing transformation and of their integrability conditions (which appear in our treatment as additional requirements for renormalizability) becomes then much simpler; a fact which we are finding useful in the study, now in progress, of special theories. The investigation of the Lee model along these lines, to be published shortly ([7]), gives results in complete agreement with those well known from the study of it in momentum space.

1·2. – We shall refer throughout in this work to electrodynamics, as was done before: it is understood, however, that the same, or very similar, considerations can be made also for all other renormalizable theories.

([5]) The connection is best seen by remarking that the problem is, no matter what technique is used, that of solving *correctly* hyperbolic equations: all the methods mentioned here do this, if properly handled; our approach is intended to preserve this generality, so that each special question may be treated with the technique best adapted to it without having to give a new proof for each new case.

([6]) J. VALATIN: *Proc. Roy. Soc.* (*London*), A **222**, 93 and 228 (1954); **225**, 535 (1954): **226**, 254 (1954); M. RIESZ: *Acta Math.*, **81**, (1949); Riesz's method has been often applied incorrectly.

([7]) E. R. CAIANIELLO and S. OKUBO: to be published in *Nuovo Cimento*.

For the notation we refer to I and II; most of the present analysis can be understood, though, starting simply from our formal definition of finite-part integral (I-44)

$$\int F(\xi, \xi_1, ..., \xi_n)\, \mathrm{d}\xi_i = \int_\xi F\, \mathrm{d}\xi_i + D^i_\xi F\,,$$

where the first term at r.h.s. denotes a well-defined expression, calculated so as to remove the divergent contributions which may occur when ξ_i is on the light-cone of ξ (*i.e.* «*confluent*» with ξ), and $D^i_\xi F$, the infinite part, is the difference between the l.h.s. and the term just described.

The reader will notice by himself both similarities and differences between the present analysis and the well-known treatment by A. SALAM ([8]).

2. – Extension of previous results.

2·1. – We wish, first of all, to establish a combinatorial formula more general than (I-45), which was the starting point of the analysis presented in I and II. We re-write (I-45) as

$$(1) \qquad \int \mathrm{d}\xi_n \ldots \int \mathrm{d}\xi_1 F(\xi; \xi_1, ..., \xi_n) = \left[\int_\xi \mathrm{d}\xi_n + D^n_\xi\right] \ldots \left[\int_\xi \mathrm{d}\xi_1 + D^1_\xi\right] F =$$

$$= \sum_{h=0}^{n} \binom{n}{h} D^n_\xi \ldots D^{n-h+1}_\xi \int_\xi \mathrm{d}\xi_{n-h} \ldots \int_\xi \mathrm{d}\xi_1 F\,,$$

and remind that (1) is a consequence of the assumption that the finite-part integrals satisfy the condition (I-46)

$$(2) \qquad \int_\xi \mathrm{d}\xi_h D^k_\xi = D^h_\xi \int_\xi \mathrm{d}\xi_k\,,$$

under hypothesis of symmetry of the integrand in its integration variables.

We remark that it suffices, for the validity of all the combinatorial work of I and II and of the formulae which will be derived here, that the integrands $F(\xi; \xi_1, ..., \xi_n)$ be symmetric functions only of the arguments $\xi_1, ..., \xi_n$; the argument ξ need not share this property (we omit the explicit mention of

([8]) A. SALAM: *Phys. Rev.*, **82**, 217 (1951); **84**, 426 (1951).

external variables). This remark is important only in the study of the derivatives of kernels with respect to masses, which was not touched upon in I and II, nor will be undertaken here.

The extension of (1) which we propose to obtain is related to the replacement of (2) with the much more general condition ($h > m$)

$$(3) \qquad D_\xi^{h,\,h-1,\,\ldots,\,m+1}\int_\xi \mathrm{d}\xi_m = \int_\xi \mathrm{d}\xi_h\, D_\xi^{h-1,\,\ldots,\,m} + \\ - D_\xi^{h,\,\ldots,\,m+1}\cdot D_\xi^{m} + D_\xi^{h}\cdot D_\xi^{h-1,\,\ldots,\,m} + D_\xi^{h,\,h-1,\,\ldots,\,m},$$

where, for the sake of simplicity, we forgo writing the symmetric functions $F(\xi;\,\xi_1 \ldots \xi_n)$ on which these operators act. (3) covers, indeed, all the cases in which we may be possibly interested; its mathematical and physical implications are analyzed later, as it is important to settle the combinatorics first.

We wish to prove that, when (3) replaces (2), (1) changes to

$$(4) \qquad \int \mathrm{d}\xi_n \ldots \int \mathrm{d}\xi_1\, F(\xi;\,\xi_1,\ldots,\xi_n) - \sum_{h=0}^{n}\binom{n}{h}\hat{\mathscr{D}}_\xi^{n,\,\ldots,\,n-h+1}\int_\xi \mathrm{d}\xi_{n-h}\ldots\int_\xi \mathrm{d}\xi_1\, F,$$

which can be written, alternatively, in the form

$$(5) \qquad \int \mathrm{d}\xi_n \ldots \int \mathrm{d}\xi_1\, F = \sum_{h=0}^{\infty}\binom{n}{h}\int_\xi \mathrm{d}\xi_n \ldots \int_\xi \mathrm{d}\xi_{n-h+1}\mathscr{D}_\xi^{n-h\,\ldots,\,1}F,$$

where

$$(6) \qquad \hat{\mathscr{D}}_\xi^{n,\,\ldots,\,l} = \sum_{k=0}^{n-l}(-1)^{n-l+k}\binom{n-l}{k} D_\xi^{n,\,\ldots,\,l+k}\hat{\mathscr{D}}_\xi^{l+k-1,\,\ldots,\,l},$$

$$(7) \qquad \mathscr{D}_\xi^{n,\,\ldots,\,l} = \sum_{k=0}^{n-l}\binom{n-l}{k}\mathscr{D}_\xi^{n,\,\ldots,\,l+k+1}D_\xi^{l+k,\,\ldots,\,l},$$

with

$$(8) \qquad \hat{\mathscr{D}}_\xi^{h} = \mathscr{D}_\xi^{h} = D_\xi^{h}; \qquad \hat{\mathscr{D}}_\xi^{h-1,h} = \mathscr{D}_\xi^{h-1,h} = 1.$$

(6) and (7) are written in recursive form for brevity: it should be understood that the l.h.s.'s are properly defined only when total iteration is carried through at the r.h.s.'s. All superscripts in a given D_ξ, $\hat{\mathscr{D}}_\xi$, or $\mathscr{D}_\xi$ occur always in decreasing magnitude from left to right, with the formal exception of the case $n = 1$, for which we define tham as in (8).

The proofs are easily obtained by induction. We prove first, from (6) and (7), the lemmas

$$\int_{\xi} \mathrm{d}\xi_{n+1} \hat{\mathscr{D}}_{\xi}^{n,\ldots,l} = \hat{\mathscr{D}}_{\xi}^{n+1,\ldots,l+1} \int_{\xi} \mathrm{d}\xi_l + \left(\hat{\mathscr{D}}_{\xi}^{n+1,\ldots,l} - \hat{\mathscr{D}}_{\xi}^{n+1} \hat{\mathscr{D}}_{\xi}^{n,\ldots,l}\right), \tag{9}$$

and

$$\mathscr{D}_{\xi}^{n,\ldots,l+1} \int_{\xi} \mathrm{d}\xi_l = \int_{\xi} \mathrm{d}\xi_n \mathscr{D}_{\xi}^{n-1,\ldots,l} + \left(\mathscr{D}_{\xi}^{n,\ldots,l} - \mathscr{D}_{\xi}^{n,\ldots,l+1} \mathscr{D}_{\xi}^{l}\right). \tag{10}$$

Both formulae are true for $n = 1$, because of (3), (6) and (7); assuming them to be true up to a given $n > 1$, we find from (6), (3) and (9)

$$\int_{\xi} \mathrm{d}\xi_{n+1} \hat{\mathscr{D}}_{\xi}^{n,\ldots,l} = \sum_{k=0}^{n-l} (-1)^{n-l+k} \binom{n-l}{k} \Big\{ D_{\xi}^{n+1,\ldots,l+k+1} \int_{\xi} \mathrm{d}\xi_{l+k} + \tag{11}$$

$$+ D_{\xi}^{n+1,\ldots,l+k+1} D_{\xi}^{l+k} - D_{\xi}^{n+1} D_{\xi}^{n,\ldots,l+k} - D_{\xi}^{n+1,\ldots,l+k} \Big\} \hat{\mathscr{D}}_{\xi}^{l+k-1,\ldots,l} =$$

$$= \hat{\mathscr{D}}_{\xi}^{n+1,\ldots,l+1} \int_{\xi} \mathrm{d}\xi_l + \sum_{k=0}^{n-l} (-1)^{n-l+k} \binom{n-l}{k} \Big\{ D_{\xi}^{n+1,\ldots,l+k+1} [\hat{\mathscr{D}}_{\xi}^{l+k,\ldots,l} +$$

$$- \hat{\mathscr{D}}_{\xi}^{l+k} \hat{\mathscr{D}}_{\xi}^{l+k-1,\ldots,l}] + [D_{\xi}^{n+1,\ldots,l+k+1} D_{\xi}^{l+k} - D_{\xi}^{n+1} D_{\xi}^{n,\ldots,l+k} +$$

$$- D_{\xi}^{n+1,\ldots,l+k}] \hat{\mathscr{D}}_{\xi}^{l+k-1,\ldots,l} \Big\} = \hat{\mathscr{D}}_{\xi}^{n+1,\ldots,l+1} \int_{\xi} \mathrm{d}\xi_l +$$

$$+ \sum_{k=0}^{n-l} (-1)^{n-l+k} \binom{n-l}{k} \Big\{ D_{\xi}^{n+1,\ldots,l+k+1} \hat{\mathscr{D}}_{\xi}^{l+k,\ldots,l} - \hat{\mathscr{D}}_{\xi}^{n+1} D_{\xi}^{n,\ldots,l+k} \hat{\mathscr{D}}_{\xi}^{l+k-1,\ldots,l} +$$

$$- D_{\xi}^{n+1,\ldots,l+k} \hat{\mathscr{D}}_{\xi}^{l+k-1,\ldots,l} \Big\} = \hat{\mathscr{D}}_{\xi}^{n+1,\ldots,l+1} \int_{\xi} \mathrm{d}\xi_l - \hat{\mathscr{D}}_{\xi}^{n+1} \hat{\mathscr{D}}_{\xi}^{n,\ldots,l} +$$

$$+ \sum_{h=1}^{n-l+1} (-1)^{n+1-l+h} \binom{n-l}{h-1} D_{\xi}^{n+1,\ldots,l+h} \hat{\mathscr{D}}_{\xi}^{l+h-1,\ldots,l} +$$

$$+ \sum_{h=0}^{n-l} (-1)^{n+1-l+h} \binom{n-l}{h} D_{\xi}^{n+1,\ldots,l+h} \hat{\mathscr{D}}_{\xi}^{l+h-1,\ldots,l} = \hat{\mathscr{D}}_{\xi}^{n+1,\ldots,l+1} \int_{\xi} \mathrm{d}\xi_l +$$

$$- \hat{\mathscr{D}}_{\xi}^{n+1} \hat{\mathscr{D}}_{\xi}^{n,\ldots,l} + \sum_{h=0}^{n+1-l} (-1)^{n+1-l+h} \binom{n+1-l}{h} D_{\xi}^{n+1,\ldots,l+h} \hat{\mathscr{D}}_{\xi}^{l+h-1,\ldots,l};$$

the latter expression is identical with (9): q.e.d.: The proof of (10) is entirely analogous.

Suppose now that (4) is true up to a given n. Then (using for brevity the convention $1 \equiv \int_{\xi} \mathrm{d}\xi_0$)

$$(12)\qquad \left(\int_{\xi} \mathrm{d}\xi_{n+1} \cdot D_{\xi}^{n+1}\right) \int \mathrm{d}\xi_n \dots \int \mathrm{d}\xi_1 = \int_{\xi} \mathrm{d}\xi_{n+1} \sum_{h=0}^{n} \binom{n}{h} \widehat{\mathscr{D}}_{\xi}^{n, \dots, n-h+1} \int_{\xi} \mathrm{d}\xi_{n-h} \dots \int_{\xi} \mathrm{d}\xi_1 \cdot$$

$$- D_{\xi}^{n+1} \sum_{h=0}^{n} \binom{n}{h} \widehat{\mathscr{D}}_{\xi}^{n, \dots, n-h+1} \int_{\xi} \mathrm{d}\xi_{n-h} \dots \int_{\xi} \mathrm{d}\xi_1 = \sum_{h=0}^{n} \binom{n}{h} \widehat{\mathscr{D}}_{\xi}^{n+1, \dots, n-h+2} \int_{\xi} \mathrm{d}\xi_{n-h+1} \dots \int_{\xi} \mathrm{d}\xi_1 +$$

$$+ \sum_{h=0}^{n} \binom{n}{h} [\widehat{\mathscr{D}}_{\xi}^{n+1, \dots, n-h+1} + D_{\xi}^{n+1} \widehat{\mathscr{D}}_{\xi}^{n, \dots, n-h+1} - \widehat{\mathscr{D}}_{\xi}^{n+1} \cdot \widehat{\mathscr{D}}_{\xi}^{n, \dots, n-h+1}] \int_{\xi} \mathrm{d}\xi_{n-h} \dots \int_{\xi} \mathrm{d}\xi_1 =$$

$$= \sum_{h=0}^{n+1} \binom{n+1}{h} \widehat{\mathscr{D}}_{\xi}^{n+1, \dots, n-h+2} \int_{\xi} \mathrm{d}\xi_{n-h+1} \dots \int_{\xi} \mathrm{d}\xi_1 , \qquad \text{q.e.d.}$$

The proof of formula (5) is entirely similar.

We find, in conclusion, that the replacement of condition (2) with condition (3), which is the most general we may conceive, has the only effect of changing $D_{\xi}^{n} \dots D_{\xi}^{n-h+1}$ in (1) with $\widehat{\mathscr{D}}_{\xi}^{n, \dots, n-h+1}$ in (4) and $\mathscr{D}_{\xi}^{n, \dots, n-h+1}$ in (5); but for these two changes, all combinatorial analysis of I and II still applies, because of the symmetry of the integrands.

The central formula of our study of renormalization, (I-47), becomes after the change of (2) with (3)

$$(13)\qquad \frac{\partial}{\partial \lambda} K \left(\begin{matrix} x_1 \dots x_{N_0} \\ y_1 \dots y_{N_0} \end{matrix} \middle| t_1 \dots t_{P_0} \right) =$$

$$= \sum_{h=0}^{\infty} \frac{\lambda^h}{h!} \int \mathrm{d}\xi \sum \gamma^{\xi} \widehat{\mathscr{D}}_{\xi}^{h, \dots, 1} \gamma^1 \dots \gamma^h K^{(\xi)} \left(\begin{matrix} x_1 \dots x_{N_0} & \xi\xi_1 \dots \xi_h \\ y_1 \dots y_{N_0} & \xi\xi_1 \dots \xi_h \end{matrix} \middle| \begin{matrix} \xi\xi_1 \dots \xi_h & t_1 \dots t_{P_0} \end{matrix} \right)$$

and similar results hold for the derivatives of the kernels with respect to the other parameters of the theory. It is then easy to see, since (4) and (5) imply (all integrands are symmetric in $\xi_1 \dots \xi_m \xi'_1 \dots \xi'_n$) that:

$$(14)\qquad \sum_{m+n=p} \frac{1}{m!\, n!} \widehat{\mathscr{D}}_{\xi}^{m, \dots, 1} \int_{\xi} \mathrm{d}\xi_n^1 \dots \int_{\xi} \mathrm{d}\xi_1^1 = \sum_{m+n=p} \frac{1}{m!\, n!} \int_{\xi} \mathrm{d}\xi'_n \dots \int_{\xi} \mathrm{d}\xi'_1 \mathscr{D}_{\xi}^{m, \dots, 1} ,$$

that the considerations of I and II apply *in toto* also for prescriptions for the calculation of finite-part integrals satisfying condition (3), provided the di-

vergent cores defined in II by (II-11) and (II-12) are instead constructed from

$$(15)\qquad L^{(\lambda)}\left(\begin{matrix}\overset{\circ}{u}_1 \dots \overset{\circ}{u}_n \xi \\ \overset{\circ}{v}_1 \dots \overset{\circ}{v}_n \xi\end{matrix}\middle| \xi \overset{\circ}{w}_1 \dots \overset{\circ}{w}_P\right) =$$

$$= \sum_{M(P+1)} \frac{\lambda^M}{M!} \mathscr{D}_\xi^{M, \dots, 1} \sum \gamma^1 \dots \gamma^M \begin{pmatrix} \overset{\circ}{u}_1 \dots \overset{\circ}{u}_n & \xi\xi_1 \dots \xi_M \\ \overset{\circ}{v}_1 \dots \overset{\circ}{v}_n & \xi\xi_1 \dots \xi_M \end{pmatrix} [\xi\xi_1 \dots \xi_M \quad t_1 \dots t_P],$$

and similarly for those relative to the other parameters of the theory.

2·2. – A few formal remarks, intended to give a better intuitive grasp of the results mentioned above, may be in order.

It can be verified easily that (6) and (7) can be written symbolically as

$$(6')\qquad (\underset{\sim}{\hat{\mathscr{Q}}}_\xi)^n = P_{\mathscr{D}\hat{\mathscr{D}}} \{\underset{\sim}{D}_\xi [\underset{\sim}{\hat{\mathscr{Q}}}_\xi - \underset{\sim}{D}_\xi]^{n-1}\},$$

$$(7')\qquad (\underset{\sim}{\mathscr{Q}}_\xi)^n = P_{\mathscr{Q}D} \{\underset{\sim}{D}_\xi [\underset{\sim}{\mathscr{Q}}_\xi + \underset{\sim}{D}_\xi]^{n-1}\},$$

where P_{ab} denotes that any product of the operators $\underset{\sim}{a}$ and $\underset{\sim}{b}$ must be ordered by putting always $\underset{\sim}{a}$ at the left of $\underset{\sim}{b}$ (thus, $P_{ab}\{\exp[\underset{\sim}{a}+\underset{\sim}{b}]\} = \exp[\underset{\sim}{a}]\exp[\underset{\sim}{b}]$), and the powers of the symbols $\underset{\sim}{D}_\xi$, $\underset{\sim}{\mathscr{Q}}_\xi$, $\underset{\sim}{\hat{\mathscr{Q}}}$ are defined so as to match (6) and (7).

Likewise, if one chooses to write formula (13) below as

$$(13')\qquad \frac{\partial K_{N_0 P_0}}{\partial \lambda} = \int \mathrm{d}\xi \sum \gamma^\xi \exp[\lambda \underset{\sim}{\mathscr{Q}}]_\xi \underset{\sim}{K}^{(\xi)} \left(\begin{matrix} x_1 \dots x_{N_0} & \xi \\ y_1 \dots y_{N_0} & \xi \end{matrix}\middle| \xi t_1 \dots t_{P_0}\right),$$

where $\underset{\sim}{\hat{\mathscr{Q}}}_\xi$ is the operator defined by (6') (which could easily be written explicitly by using functional derivative), one finds:

$$\frac{\partial}{\partial \lambda} \exp[\lambda \underset{\sim}{\hat{\mathscr{Q}}}_\xi] = \underset{\sim}{D}_\xi \exp[-\lambda \underset{\sim}{D}_\xi] \exp[\lambda \underset{\sim}{\hat{\mathscr{Q}}}],$$

etc. We do not insist on this possible algebrization of our procedure, although it may prove quite interesting, because it is not essential for our present purposes. We just notice that, assuming that all the operators we are handling in this section commute (which is far from true), appropriate definitions of the symbolic multiplications of these operators would yield

$$\exp[\lambda \underset{\sim}{\hat{\mathscr{Q}}}_\xi] = \exp[1 - \exp[-\lambda \underset{\sim}{D}_\xi]]; \qquad \exp[\lambda \underset{\sim}{\mathscr{Q}}_\xi] = \exp[1 - \exp[\lambda \underset{\sim}{D}_\xi]].$$

This last remark may give a rough idea of the structure of the objects with which we are dealing; a precise statement of these properties would be useful in the study of the structural changes that a finite renormalization performs into the kernels.

3. – Finite-part integrals in configuration space.

3·1. – In the previous Part we have dealt with the situation that arises when computations are performed with finite-part integrals satisfying a condition of type (3). This is important, because a non-perturbative solution of the branching equations—aside from the strictly mathematical problems posed by the use of finite-part integrals, which we do not attempt to discuss here—requires in general a prescription of this nature, which is clearly intended to take care of the possibility that a divergence may arise only after a number of divergenceless integrations are performed: $D_{\xi}^{k,\dots,1}$ contains only, and all, the divergences which appear in the k-fold integration over $\xi_1, \dots, \xi_k$ because of confluences of these variables with ξ (but not among themselves alone) and does not contain divergences due to confluences of order lower or higher than k. (This becomes obvious if formula (3) is written in terms of ordinary integrals.)

It is however convenient in many questions, expecially those pertaining to the study of the renormalizability of a theory and of the Lie equations of its renormalization group, or in the investigation of perturbative expansions, to use subtractive prescriptions which fulfil the simpler prescription (2); this can be achieved, in general, only if *all* the required integrations and integrands are displayed beforehand, as it happens with perturbative computations. This is possible because, as we shall see, it is then our privilege to decide at which stage a term which will eventually give rise to divergences should be subtracted.

Rather than treating the integrand factors as prima-facie distributions, it is convenient for us to go back to first principles and remind that distributions are symbols indicating that, *after* some integrations are performed upon regular functions—which depend upon parameters and reduce to the corresponding distributions only when these reach their limits—the said parameters are to be taken to their limits. It is to our advantage, here, to adhere to this original definition [9]: all integrand terms shall be understood to contain also regularizing parameters (which we do not write explicitly, it being relatively unimportant in which way this auxiliary regularization is made), and

[9] The same attitude is adopted in ref. [4], to which we refer the reader for a thorough mathematical analysis.

all operations of subtraction or renormalization shall be made upon integrands thus regularized. Only *after* all the operations that are necessary for the evaluation of finite-part integrals are performed shall the regularizing parameters be made to tend to their limits, and the subtracted or otherwise renormalized terms assume their infinite or indeterminable values. It is of course clear that the calculation of finite-part integrals requires the *a priori* knowledge of the nature of the singularities that would arise if ordinary integrals were used in their place. The properties of symmetry of the integrands avoid then, among other things, questions as to the legitimacy of multiplying distributions.

3·2. – We propose to discuss a particular subtractive procedure in configuration space, which applies only to symmetric integrands and removes therefrom all, and only, the divergences arising from ξ-confluences; it obeys the mathematical *principle of permanence*, *i.e.* it does not modify integrals which converge in the ordinary sense. We emphasize that this procedure is preferred here only for reasons of methodological convenience, and that infinitely many procedures are admissible and « equivalent » (we shall use this term henceforth to denote that the only difference may consists in a finite renormalization). Let

$$\eta_i^\mu = \xi_i^\mu - \xi^\mu , \qquad \eta_{ij}^\mu = \xi_i^\mu - \xi_j^\mu , \qquad \eta = \sqrt{|\eta^2|} . \tag{16}$$

Denote, for short, an integrand as

$$F(\xi; \xi_1, ..., \xi_n) = \sum_{(\alpha)} \frac{1}{\eta_1^{\alpha_1} ... \eta_{n-1,n}^{\alpha_{n-1,n}}} \mathcal{C}_{(\alpha)}(\xi; \xi_1, ..., \xi_n) , \tag{17}$$

where only the integration variables are explicitly indicated; each term of the sum at r.h.s. comes from the expansion of pfaffians and hafnians [1], *i.e.* corresponds to some Feynman graph (α); the summation is extended only over connected graphs (disconnected graphs are, as a matter of fact, eliminated in electrodynamics *ipso facto* by this prescription, with the exception of some vacuum parts; in any case, however, they can be eliminated with purely algebraic devices, *e.g.* by changing the scale of the variable ξ and then that of all the variables in the factors of each term of the integrand which are affected by this change: only disconnected graphs contain factors which remain unaffected. We shall refer to graphs in the following discussion only for verbal convenience, but shall use nowhere the techniques which are traditionally implied by this terminology).

F is a symmetric function of $\xi_1, ..., \xi_n$, so that for each graph (α) (17) contains all those of the same topological structure which are required for the overall symmetry.

(17) is intended to exhibit explicitly the maximum singularity that characterizes each graph (α) in the product of factors $\eta^{-\alpha}$, where the exponents α_i, α_{ij} are such, that the accompanying factors $g_{(\alpha)}(\xi; \xi_1, ..., \xi_n)$ are finite and non-vanishing holomorphic functions when any number of the variables $\xi_1, ..., \xi_n$ are confluent with ξ; this property is secured by the assumed auxiliary regularization of the integrand (see end of preceding section). Their possible values are, in electrodynamics, 0, 2, 3, 5, 6, 8.

We intend to remove all, and only, the divergences which arise upon integration of (17) because of confluences of $\xi_1, ..., \xi_n$ with ξ: a confluence of order k is characterized by $\eta_{i_1} = \eta_{i_2} = ... = \eta_{i_k} = 0$, all other $\eta_i \neq 0$. In a neighborhood of the origin each function $g_{(\alpha)}$ admits the expansion

$$g_{(\alpha)}(\xi; \xi_1, ... \xi_n) = \sum_{\text{all } h_\mu \geqslant 0} A^{(\alpha)}_{h_{\mu_1} ... h_{\mu_n}} (\eta_1^{\mu_1})^{h_{\mu_1}} ... (\eta_n^{\mu_n})^{h_{\mu_n}}, \tag{18}$$

with $A^{(\alpha)}_{0, ..., 0} \neq 0$. The sum in (18) cannot, of course, be exchanged with the integration over $\xi_1, ..., \xi_n$; for all our purposes, we shall use (18) with the understanding that it denotes a finite number of as many terms as convenient, plus a remainder expressed in closed form. Even so, spurious divergences of infrared nature may arise for large values of $\eta_1, ..., \eta_n$, if things are not handled with sufficient care; we shall soon return to this point.

We have to consider the behaviour around the origin of the integral of a product like

$$\frac{(\eta_1^{\mu_1})^{h_{\mu_1}} ... (\eta_n^{\mu_n})^{h_{\mu_n}}}{\eta_1^{\alpha_1} ... \eta_{n-1,n}^{\alpha_{n-1,n}}}. \tag{19}$$

It might at first appear that, since all the η^α are powers of Lorentz squares, integrals such as $\int d^4\eta\, \eta^{-\alpha} (\eta^\mu)^{h_\mu}$ might diverge in the origin also if h_μ has large values; this is not so, however, because of the causal character of the propagators from which all such terms originate. One could prove this by going over to Euclidean metric, but it is far simpler to remark that $(\eta^2 = (\boldsymbol{\eta})^2 - \eta_0^2)$

$$\int d^4\eta\, \eta^{-\beta} (\eta^\mu)^{h_\mu} = i^{\beta/2} \Gamma\left(\frac{\beta}{2}\right) \int d^4\eta\, (\eta^\mu)^{h_\mu} \int_0^\infty d\alpha\, \exp[i\alpha\eta^2 - \varepsilon\alpha]\, \alpha^{(\beta/2)-1} \sim \tag{20}$$

$$\sim \begin{cases} 0 & (h_\mu \text{ odd}) \\ \displaystyle\int_0^\infty d\alpha\, \exp[-\varepsilon\alpha]\ \alpha^{((\beta - h_\mu)/2)-3} & (h_\mu \text{ even}). \end{cases}$$

(20) shows also that (again introducing the $i\varepsilon$ which gives causal behavior)

$$\int d^4\eta\, (\eta^\mu)^{h_\mu} \eta^{-\beta} \exp[iC\eta^2], \qquad C > 0, \tag{21}$$

is convergent for $\eta \to \infty$ ($\alpha \to 0$ in (20)). Factors like (21)— or some such— are required in a subtraction of terms like (19), so that the singularities at the origin may be removed without incurring in the spurious singularities mentioned above [10]. These factors also secure that causality is respected, as is apparent from (20) and (21).

3·3. – The first requirement we impose is that, after subtractions, *each* term of (17), where use is made of (18), be rid of divergences due to ξ-confluences for *all* possible ordering $i_1, \ldots, i_n$ of the integrations performed upon these variables. This condition is clearly achieved if the summation $\sum_{\text{all } h\mu \geqslant 0}$ in (18) is replaced with $\sum'_{\text{all } h_\mu}$, defined so as to range only over the values of $h_{\mu_1}, \ldots, h_{\mu_n}$ which satisfy *all* of the following inequalities:

$$(A_\alpha) \qquad \left\{ \begin{array}{l} h_{\mu_{i_1}} \geqslant a_{i_1} \\ h_{\mu_{i_1}} + h_{\mu_{i_2}} \geqslant a_{i_1 i_2}, \\ \cdots\cdots\cdots\cdots \\ h_{\mu_{i_1}} + \ldots + h_{\mu_{i_n}} \geqslant a_{i_1 \ldots i_n} \end{array} \right. \qquad \text{(all permutations } i_1 \ldots i_n\text{)}.$$

This system contains all the inequalities which obtain for all possible orderings $i_1 \ldots i_n$ of the integrations; we shall refer to it as «system A_α» when it is important to specify that it relates to the graph (α), and shall call, for short, «criterion A» (or B, etc.) a subtractive procedure of this type. In (A_α)

$$(22) \qquad a_{i_1 \ldots i_k} = \text{Max}\left[0; \sum_{r=1}^{k} \alpha_{i_r} + \sum_{r,s=1}^{k} \alpha_{i_r i_s} - (4k-1)\right].$$

The k-th inequality of A_α secures, of course, only the convergence around ξ of the k-fold integral on $\xi_{i_1}, \ldots, \xi_{i_k}$. Clearly, $a_{i_1 \ldots i_n} = a_{1, \ldots, n}$. The part which is subtracted is defined as $\sum''_{\text{all } h_\mu}$ in the equality

$$(23) \qquad \sum_{\text{all } h_\mu \geqslant 0} = \sum'_{\text{all } h_\mu} + \sum''_{\text{all } h_\mu}.$$

Factors like $\exp[iA\eta^2]$ in (21) will secure that no spurious infrared divergences are introduced; $\sum'$ and $\sum''$ shall be understood to be defined so as to take

[10] We note that our treatment of ultraviolet divergences does not mix them with divergence of infrared nature; the same cannot be said of several current discussions made in momentum space. This question becomes relevant if one attaches significance to the magnitude of renormalization constants.

due care of the introduction of such factors, which we shall omit noting explicitly in the following. (A change of the set of values C into another such set would only amount to a finite renormalization.)

The system A is superabundant. It can be reduced in a unique fashion to a set of mutually independent inequalities, which we name *system* B, by suppressing all the inequalities which follow from others of the set, or in which $a_{i_1 \dots i_k} = 0$. We have to concern ourselves only with system B; we may either impose that $\sum'$ be restricted by the inequalities B_α, or choose to work with a *minimal solution* of B_α of type

$$h_1 \geqslant \beta_1; \quad h_2 \geqslant \beta_2; \quad \dots; \quad h_n \geqslant \beta_n \tag{24}$$

(such, that is, that values $h_r \geqslant \beta_r - 1$ would not satisfy B_α). We shall refer to a set of inequalities such as (24) as *system* C; it must be emphasized, however, that several non-identical systems C_α may be found for a same graph (α), so that a careless handling of criterion C may give rise to ambiguities: a well-known situation, which however can cause no trouble if proper care is taken of the over-all symmetry of (17), as will be shown in Section 5.

3·4. – It is a straightforward matter to determine the maximum permissible number of inequalities in systems B_α or C_α. For a fixed graph (α) of (17), and for a fixed order of integration $i_1, \dots, i_n$, one has, clearly

$$a_{i_1 \dots i_k} = 3I_f^{(i_1 \dots i_k)} + 2I_b^{(i_1 \dots i_k)} - (4k-1), \tag{25}$$

where $I_f^{(i_1 \dots i_k)}$ is the number of electron propagators having both arguments from among $\xi, \xi_{i_1}, \dots, \xi_{i_k}$ (internal electron lines), and $I^{(i_1 \dots i_k)}$ is similarly defined for photon. Since

$$\begin{cases} E_f^{(i_1 \dots i_k)} + 2I_f^{(i_1 \dots i_j)} = 2(k+1), \\ E_b^{(i_1 \dots i_k)} + 2I_b^{(i_1 \dots i_k)} = k+1, \end{cases} \tag{26}$$

where $E_{f,b}^{(i_1 \dots i_k)}$ denotes the number of electron or photon lines having only one argument from among $\xi, \xi_{i_1}, \dots, \xi_{i_k}$ (external lines), we find

$$a_{i_1 \dots i_k} = \mathrm{Max}\,[0;\ 5 - (\tfrac{3}{2} E_f^{(i_1 \dots i_k)} + E_b^{(i_1 \dots i_k)})]. \tag{27}$$

From the structure of system A_α it is then apparent that, for any given ordering $i_1 \dots i_n$, there can be at the very worst five inequalities in the subsystem of A_α which corresponds to the chosen ordering. We count as one inequality all those that obtain by varying the indices μ of $h_{\mu_1}, \dots, h_{\mu_n}$, and write henceforth, for short, h_i in place of h_{μ_i}).

We can now prove that five is also the maximum number of inequalities in system B_ν or C_ν. Consider, as above, the subsystem of A which corresponds to a given ordering of the integrations $i_i \dots i_n$; whenever there is an effective increase at the r.h.s. from the k-th to the $(k+1)$-th inequality (which therefore is not a consequence of the previous one), this can only occur, as is apparent from (27), because of the closing of a loop (with fermion or boson lines) on a line connected with ξ, which may be in particular a path from ξ to ξ. It is clear then that, after a first electron or photon self-energy part on a line originating in ξ has given its contribution (27), all subsequent parts of the same nature on that line cannot increase the value (27); until a new, more divergent such loop is formed, or the path closes on ξ. Thus, for instance, a typical vertex part, originating in ξ with a photon line, can give an additional contribution to (27) only when—no matter after how many intermediate steps—it closes again into a photon line.

These remarks, which are fairly obvious, are relevant here because they show that the search for divergences which are due solely to ξ-confluences —which is the keystone of our method—reduces the problem essentially to that of considering only the *primitive* divergences. We are thus enabled, by an appropriate use fo combinatorics, to disentangle the rather confused situation which arises when the contributions of all possible disjoint confluences cannot be analysed further.

The system A_α can thus be ordered in subsystems, of at most five inequalities, one for each possible ordering. Choose a subsystem among those having the highest number of inequalities and fix in it an inequality; it follows from our previous remarks that all other subsystems either have at least one inequality containing one, and the same, variable in common with the fixed inequality, or have a smaller number of inequalities. Solve simultaneously all the inequalities of A thus chosen, which contain that common variable h_r, by giving h_r the maximum value found at the r.h.s. of all such inequalities: this determines the first inequality of system C_α. By iteration, after the suppression of all the inequalities of A_α which have been satisfied with this choice of h_r, one finds a system C_α containing at most five inequalities. System C_α is a solution of system B_ν, which can be shown, with the same arguments, to consist itself of at most five inequalities.

3·5. – We can now return to the original problem of the utilization of the subtraction (23), performed with system C or B, as a finite-part integration rule for perturbative expansions.

We begin with system C, and remark that, although there is in general a variety of such systems for special graphs (the same that gives rise to the well-known ambiguities of subtractive techniques in momentum space, which were first resolved by A. SALAM [8]), each permissible system C_α does perform

a correct regularization of the graph (α); the problem reduces, therefore, to fixing, more or less arbitrarily, additional rules whereby all graphs result treated in a consistent manner, so as to yield contributions which combine together in the expression (1); again, a different choice may at most imply an additional finite renormalization. We request therefore:

a) that the number of inequalities of C be minimum;

b) that the graphs of the same topological structure as (α)—which form with (α) a symmetry class of (17)—be ordered with reference to a general pre-established criterion of enumeration, which can be chosen arbitrarily, so that we may speak of a «first» graph;

c) that the h's be chosen in the inequalities of C_α with reference to a pre-established, but otherwise arbitrary, ordering of $\xi_1, \xi_2, \ldots, \xi_n$; if there is the possibility of multiple choices, the h_r to be selected is that which comes first in that ordering;

d) that all other graphs of the symmetry class of (α)—which derive from (α) by a permutation P of the variables—be treated by means of the subtraction which obtains from that chosen in *c*) for (α) by performing the *same* permutation P on the index, or indices, of C_α.

E.g., in a typical 4-th order ambiguous photon self-energy graph (α) of vertices $\xi\xi_1\xi_2\xi_3$, which we take as the first of its symmetry class, one finds that B_α reduces to

$$h_1 + h_3 \geqslant 1 \; ; \qquad h_1 + h_2 + h_3 \geqslant 3 \; .$$

The, if we fixe in *c*) the ordering ξ_1, ξ_2, ξ_3, we find that C_α reduces to $h_1 \geqslant 3$, and not $h_3 \geqslant 3$, or $h_1 + h_3 \geqslant 3$, etc.

The graph $\xi\xi_3\xi_1\xi_2$, say, obtained from (α) by means of the permutation 132, is assigned by *d*) the inequation $h_3 \geqslant 3$.

It is easy to verify that this criterion (which is specified only for the purpose of examplification) yields, if applied to (17), results which do not depend on the choices of ordering made in *b*) and in *c*), and that it satisfies condition of type (2). Write

$$(28) \qquad \int \mathrm{d}\xi_n \ldots \int \mathrm{d}\xi_1 \, F(\xi; \xi_1 \ldots, \xi_n) = \sum_{h_{\mu_1} \ldots, h_{\mu_n} \geq 0} \cdot$$

$$\cdot \int \mathrm{d}\xi_n \ldots \int \mathrm{d}\xi_1 \sum_{(\alpha)} A^{(\alpha)}_{h_{\mu_1} \ldots h_{\mu_n}} \prod_{ij=0}^{n} (\eta_{ij}^{-\alpha_{ij}}) \eta_1^{h_{\mu_1}} \ldots \eta_n^{h_{\mu_n}} \qquad (\eta_{i0} = \eta_i)$$

and define:

$$\int_{\xi} \mathrm{d}\xi_i F = \sum_{\hbar_{\mu_i} \geq \beta_i} \int \mathrm{d}\xi_i \sum_{(\alpha)} \prod_{i,j=0}^{n} A^{(\alpha)}_{\hbar_{\mu_1} \dots \hbar_{\mu_n}} (\eta_{ij}^{-\alpha_{ij}}) \eta_1^{\hbar_{\mu_1}} \dots \eta_n^{\hbar_{\mu_n}}, \tag{29}$$

$$D_i^{\xi} F = \sum_{\hbar_{\mu_i} > \beta_i} \left(\int \mathrm{d}\xi_i \sum_{(\alpha)} \prod_{i,j=0}^{n} A^{(\alpha)}_{\hbar_{\mu_1} \dots \hbar_{\mu_n}} (\eta_{ij}^{-\alpha_{ij}}) \eta_1^{\hbar_{\mu_1}} \dots \eta_n^{\hbar_{\mu_n}} \right). \tag{30}$$

Observe, besides the obvious fact that such a prescription can be meaningful only for perturbative expansions, that the condition (2) is certainly satisfied by (29) and (30) for each graph individually, regardless of symmetries: but that there is no guarantee, when ambiguities arise in the transition from system B to system C, that the same numerical quantities are subtracted from graphs having the same structure, so as to insure the validity of the fundamental formula (1) and permit, at the same time, to compute only one of the terms of a given symmetry class. That this is actually the case is secured by rules *a*) to *d*), by the symmetry of (17) and by the fact that, *after* the subtractions (which *depend* upon ordering), the orders of integrations can be freely exchanged in a neighborhood of ξ regardless of symmetry considerations (since the subtractions guaranty absolute, and therefore uniform, convergence). Call, indeed

$$\Theta^{C}_{\xi} F(\xi; \xi_1, \dots, \xi_n) = \sum_{\hbar_{\mu_1} \geq \beta_1, \dots, \hbar_{\mu_n} \geq \beta_n} \sum_{(\alpha)} \prod_{i,j=0}^{n} A^{(\alpha)}_{\hbar_{\mu_1} \dots \hbar_{\mu_n}} (\eta_{ij}^{-\alpha_{ij}}) \eta_1^{\hbar_{\mu_1}} \dots \eta_n^{\hbar_{\mu_n}} \tag{31}$$

the result of the subtraction performed with a permissible criterion C (*i.e.* satisfying *a*) to *d*)); a different choice of orderings in *b*) and *c*) will give rise to another criterion C', which can be obtained equivalently, by applying Θ^{C}_{ξ} to $F(\xi; \xi'_1 \dots, \xi'_n)$, where $\xi'_1, \dots, \xi'_n$ is a permutation of $\xi_1, \dots, \xi_n$

$$\Theta^{C'}_{\xi} F(\xi; \xi_1, \dots, \xi_n) \equiv \Theta^{C}_{\xi} F(\xi; \xi'_1, \dots, \xi'_n) ; \tag{32}$$

Clearly, because of the properties just mentioned:

$$\int \mathrm{d}\xi_n \dots \int \mathrm{d}\xi_1 \, \Theta^{C}_{\xi} F(\xi; \xi_1, \dots, \xi_n) = \int \mathrm{d}\xi_n \dots \int \mathrm{d}\xi_1 \, \Theta^{C}_{\xi} F(\xi; \xi_1, \dots, \xi_n), \tag{33}$$

In conclusion, we remark that the price to be paid for the adoption of this subtraction rule, which satisfies the very convenient condition (2), is that the study of renormalization becomes slightly more complicated than in II, because it may be unfeasible to give separate expansions of the divergent cores (after eq. (15)); one is, in fact, handling this way directly the multiple integral (1). This, however, cannot cause any serious trouble.

3·6. – In the rest of this work we discuss, on some examples, the use that can be made of the available freedom in the choice of renormalization procedures. A few words may be spent, here, on renormalization with criterion B. In this case no ambiguities arise; formally we may write, to denote the integrand after subtraction

$$\Theta_{\xi}^{B} F = (\prod_{i} \Theta_{\xi i}^{B}) F , \tag{34}$$

where $\Theta_{\xi i}^{B}$ is the projection operator corresponding to the i-th inequality of system B (which changes, of course, with the graph). Θ_{ξ}^{B} is thus the product, for each graph, of at most five operators, each of which imposes one of the restrictions of B to the sum at r.h.s. of (17), (18).

This done, a change of the integration variables—and therefore of summation indices in B—may bring (34) to the form (31), without the apparent ambiguities met before. Equivalently, one may associate to each $\Theta_{i\xi}^{B}$ a specified variable out of those that appear in the corresponding i-th equation of B_{α} for each graph (α), decide that the subtraction denoted by $\Theta_{i\xi}^{B}$ is to be performed as soon as the variable ξ_i is to be integrated upon (even if divergences will appear only when later integrations are performed), and define $\int_{\xi} \mathrm{d}\xi_i$ and D_{ξ}^{i} in analogy with (29) and (30). Considerations entirely analogous to those of the preceding section (we are in fact doing essentially the same subtraction) show then that condition (2) is again valid, and that one meets in the study of renormalization the same situation as was described at the end of that section.

4. – Concluding remarks.

4·1. – We have seen in Part III that finite-part integrals which satisfy condition (2) can be conveniently defined for perturbative expansions; it is our purpose now to compare prescriptions of this sort with integration criteria which satisfy condition (3). We expect, and shall find, no essential difference as long as the integrands are explicitly displayed, as it happens with perturbative expansions; when this is not the case, however, *e.g.* when a set of coupled equations is solved by iteration (a procedure which will ultimately yield the perturbative expansion), only prescriptions of the latter type are applicable and give rise to well-defined sets of coupled equations. The fact that equivalence of the prescriptions of either type is proved whenever those of the first type are applicable is of paramount importance, because it shows that *the branching equations, written in terms of finite-part integrals satisfying condition* (3), *can be taken as the defining equations of a correctly renormalized theory,*

regardless of the validity of perturbative techniques; a formal comparison between unrenormalized and renormalized equations will yield the relations between unrenormalized and renormalized parameters, either perturbatively or as solutions of the Lie equations of the renormalizing transformation. This statement was already made in II, in terms however of prescriptions which satisfy condition (3), and not (2), as was erroneously stated there. It is proved independently, and fully, by the fact that, with appropriate cut-offs, the unrenormalized expansions are holomorphic functions of the coupling constant in the whole complex plane ([11]), so that, if renormalization is made before these cut-offs are removed, again all procedures prove equivalent (cf. II, 5.1.)

As was already said, the equivalence of prescriptions satisfying condition (2) and (3) is an immediate consequence of the fact that, if the integrand is fully displayed, it makes no difference whether terms are subtracted only when divergences are actually met (condition (3)), or whether integrands are inspected before integrations are made and all dangerous terms are subtracted beforehand, as soon as a variable connected to system B has to be integrated upon. The only difference one can expect—aside from finer points such as were mentioned with reference to eq. (21), which do not alter equivalence—is in the timing and order in which the subtractions are made.

It is always possible to write the integrand in the form (17), (18), with the cautions that accompany the definition of (18). A typical prescription obeying condition (3) obtains if we perform first the integration on ξ_1, having care to subtract the divergences—if any—which arise because of the confluence of ξ_1 with ξ (but *not* with $\xi_2, \ldots, \xi_n$: auxiliary regularizing parameters will be needed; this situation, which is rather artificial, is only of theoretical interest, as in practice one wishes to remove *all* divergences, and not only those linked to ξ. The procedure becomes then again straightforward, as will be seen in the last section). The terms to be removed are, clearly, all those, which do not satisfy the inequality

$$h_{\mu_1} \geqslant a_1 , \tag{35}$$

(the definition of the $a_{i_1, \ldots, j_k}$ is the same as form system A). This done, we integrate next upon ξ_2, after eliminating divergences which arise upon confluence of ξ_2 with ξ; these consist in those—if any—introduced by the previous integration, plus those —if any—peculiar to the simple confluence of ξ_2 with ξ in the original integrand. This subtraction amounts therefore to removing from the original integrand all the terms which do not satisfy the *two* inequalities

$$h_{\mu_1} + h_{\mu_2} \geqslant a_{12} , \tag{36}$$

$$h_{\mu_2} \geqslant a_2 , \tag{37}$$

([11]) A. BUCCAFURRI and E. R. CAIANIELLO: *Nuovo Cimento*, **8**, 170 (1958).

this is made perspicuous if the integrand is written in the form (17), (18) as it is then apparent that the terms thus eliminated are all, and only, those which can give rise to divergences in the Laurent expansion around ξ. Continuing in this way, we see that the elimination of divergences with this method—which clearly satisfies condition (3), and will be referred to, for short, as « Laurent subtraction »—amounts to eliminating from a graph (α) all, and only, the terms which were subtracted with criterion B_α - criterion Λ_α (the last $a_{i_1,\ldots,i_n} = a_{1,\ldots,n}$). The only difference is that the subtractions are made at different stages: this account for the difference between (2) and (3); one will expect, however, equivalence rather than coincidence of the prescriptions, for the reasons mentioned in the previous paragraph: the only thing that matters, combinatorics, is identical in both cases.

4·2. – We wish to emphasize, in conclusion, that all that has been said thus far—as it should be clear from the discussion in II—can only give *necessary* conditions for renormalizability. The other proofs which are required, however, such as those pertaining to the « conditions of the second type » defined in II, are of a much more immediate nature, as they depend upon rather straightforward calculations, which are better discussed specifically for each theory. In a following work it will be shown that the present method permits a complete analysis of the Lee model, which gives results in entire agreement with the well-known ones obtained from momentum-space considerations.

An apparent paradox arises when one compares prescriptions which obey condition (2) with prescriptions which obey condition (3). In the first case, in electrodynamics, one finds no contributions from the typical vertex part (cf. II, 4.2); in the second case this contribution exists and can be computed explicitly. The answer is that, as was made evident by equations (II-26) and (II-28), the contributions from electron self-energy and vertex parts *cannot* be considered independently, the only thing that enters the Lie equations of the renormalizing transformations being their *joint* contribution, which is not altered, combinatorially, by the transition from a prescription of the first type to one of the second type. If anything, this shows the artificiality of renormalizative techniques which require the separate consideration of these two types of divergences and tend to treat them as if individually meaningful; even more so, if mutual cancellations are taken into account.

Finally, it needs hardly be emphasized that—if and once the renormalizability of a theory is proved—the actual computation of finite-part integrals need not certainly be made with the techniques we have chosen to examine here, as the simplest for purposes of demonstration. Any other criterion which satisfies the principle of permanence (this, actually, is not even necessary) and removes the infinite parts in a consistent and symmetry-respecting manner could be shown to be combinatorially equivalent to those described here; in

any such case, indeed, the discussion could be reduced to the analysis of an integrand (17) (18). The only proof that needs be made, once for all, in each case, is that the conditions of «the second type» described in II are satisfied—a fact which might becomes obvious in a Lorentz-covariant theory.

4·3. – The essential feature of our study of renormalization appears thus to be the analysis of the divergences which arise from confluences in a given point ξ; this suffices for qualitative questions, as well as for quantitative computations, after (I-48). Iteration of a subtractive procedure of type B or C would give the totally renormalized integrands of perturbative expansions; iteration of a Laurent subtraction, or of any procedure satisfying condition (3), in the way specified at the end of I-4.2, would give, more generally, complete renormalization independently of perturbative criteria.

* * *

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RIASSUNTO

Fu mostrato precedentemente che la sostituzione degli integrali ordinari con integrali a parte finita, che generalizzano una definizione originariamente data da Hadamard, dà ipso fatto equazioni di campo correttamente rinormalizzate, indipendentemente dalle tecniche perturbative, purchè si tenga conto delle proprietà di simmetria della teoria e di alcune condizioni quantitative (strettamente connesse alle condizioni euristiche di Dyson per la rinormalizzabilità) soddisfatte dalla teoria stessa. In questo lavoro noi estendiamo la classe degli integrali a parte finita prima considerata in modo da coprire tutti i casi che possano occorrere praticamente e discutiamo alcune prescrizioni possibili nello spazio delle configurazioni che soddisfano le proprietà desiderate. Si mostra come alcune prescrizioni sono più appropriate per lo studio di questioni generali, come la rinormalizzabilità di una teoria e la deduzione dell'equazione di Lie dal suo gruppo di rinormalizzazione, mentre altre sono più convenienti per il calcolo effettivo, una volta che la rinormalizzabilità sia stata provata. Questo lavoro sarà seguito fra poco da un altro in cui si mostra che questo metodo dà i risultati attesi nel caso del modello di Lie.

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Inelastic Electron Scattering and Nuclear Compressibility.

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Summary. — The form factors for the electron excitation of vibrational and rotational collective nuclear transitions are obtained assuming the nucleus is incompressible, and compared with the experimental and theoretical results of Helm [3]. The inelastic scattering of high energy electrons can be used to investigate nuclear compressibility.

Inelastic scattering of high energy electrons can be used to give detailed information about matrix elements for electromagnetic transitions in nuclei. Several calculations of this scattering have been made using the shell model [1,2]. Helm [3] has observed the scattering by several even-even nuclei, and some of the states excited are known to have collective properties. Ferrell and Visscher [2] have shown it is possible to give a shell model treatment of the electron excitation of collective vibrational states. However, it is convenient to use a phenomenological collective model, especially for heavy nuclei. By assuming some form for the collective nuclear motion, the transition charge density $Ze\varrho_t(\boldsymbol{r})$ is related to the static charge density of the ground state $Ze\varrho(r)$, which can be determined from elastic electron scattering experiments. The inelastic scattering is then calculated by using Born approximation.

The electron excitation of vibrational transitions has been treated [4] by assuming that the collective motion in the nucleus is incompressible and irro-

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(1) G. Morpurgo: *Nuovo Cimento*, **3**, 430 (1956); D. G. Ravenhall: unpublished; L. J. Tassie: *Nuovo Cimento*, **5**, 1497 (1957).

(2) R. A. Ferrell and W. M. Visscher: *Phys. Rev.*, **104**, 475 (1956).

(3) R. H. Helm: *Phys. Rev.*, **104**, 1466 (1956).

(4) L. J. Tassie: *Austr. Journ. Phys.*, **9**, 407 (1956); **11**, 481 (1958).

tational. However, in this treatment the finite size of the proton was neglected, and the effect of this will now be considered.

The nuclear charge density operator is

$$Ze\varrho_{\mathrm{op}}(\boldsymbol{r}) = \sum_{\text{protons}} e f_{\mathrm{p}}(\boldsymbol{r} - \boldsymbol{r}_{\mathrm{p}}) , \tag{1}$$

where f_{p} describes the distribution of charge within the proton. The matrix element of

$$ZD_{\mathrm{op}}(\boldsymbol{r}) = \sum_{\text{protons}} \delta(\boldsymbol{r} - \boldsymbol{r}_{\mathrm{p}}) \tag{2}$$

gives $ZD(r)$, the density of protons in the nucleus, or $ZD_t(\boldsymbol{r})$, the transition density. We assume that the proton is rigid, and that the motion of the centres of the proton is incompressible and irrotational. Then the treatment of reference [4] is easily modified and we obtain for electron excitation of electric 2^L-pole nuclear vibrations

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\omega} = \left(\frac{\mathrm{d}\sigma}{\mathrm{d}\omega}\right)_{\mathrm{p}} F_L^2 , \tag{3}$$

where $(\mathrm{d}\sigma/\mathrm{d}\omega)_{\mathrm{p}}$ is the point charge scattering and F_L is the nuclear form factor

$$F_L(q) = \frac{2\pi^{\frac{1}{2}} Q_{L0}}{(2L+1)^{\frac{1}{2}} \langle r^{2(L-1)} \rangle_{D_0}} I_L(q) F_{\mathrm{p}}(q) , \tag{4}$$

where

$$\left| \begin{aligned} Q_{L0} &= Ze \int r^L Y_{L0} \varrho_t(\boldsymbol{r}) \, \mathrm{d}^3\boldsymbol{r} \\ &= Ze \int r^L Y_{L0} D_t(\boldsymbol{r}) \, \mathrm{d}^3\boldsymbol{r} , \end{aligned} \right. \tag{5}$$

is the transition 2^L-pole moment, and

$$I_L(q) = (-1)^{L+1} q^{L-1} \left(\frac{\mathrm{d}}{\mathrm{d}q} q^{-1}\right)^{L-2} \frac{\mathrm{d}}{\mathrm{d}q} F_{D,e}(q) . \tag{6}$$

F_{p} is the elastic scattering form factor of the proton, and

$$F_{D,e}(q) = 4\pi \int j_0(qr) D_0(r) r^2 \, \mathrm{d}r , \tag{7}$$

would be the form factor for elastic scattering by the nucleus if the protons were point charges. $\hbar q$ is the momentum transfer.

The form factor obtained by FERRELL and VISSCHER [2] for the excitation of a 2^+ oscillation of a harmonic oscillator nucleus is of the form (6) with $L = 2$. The same correction for the finite size of the proton can be made in their case.

For deformed nuclei it is necessary to consider the excitation of rotational levels. The matrix element for the density of protons is given by [5,6] ($I_0 \neq \frac{1}{2}$)

$$(8)\quad D(\boldsymbol{r}; I_0 M_0 \to IM) = D_0(r)\,\delta_{II_0}\delta_{MM_0} + [(2I+1)(2I_0+1)]^{\frac{1}{2}}(4\pi/5)^{\frac{1}{2}} \cdot \begin{pmatrix} I & I_0 & 2 \\ I & -I_0 & 0 \end{pmatrix} (-1)^{M-I_0} D_2(r) \sum_m \begin{pmatrix} I & I_0 & 2 \\ M & -M_0 & -m \end{pmatrix} Y_{2m}(\theta, \varphi)\,,$$

when the proton density relative to rotating axes is

$$(9)\qquad D(\boldsymbol{r}) = D_0(r) + D_2(r)\,P_2(\cos\theta')\,.$$

The form factor is

$$(10)\qquad F(q) = F_p(q)\int \exp\,[i\boldsymbol{q}\cdot\boldsymbol{r}]\,D(\boldsymbol{r}; I_0 M_0 \to IM)\,\mathrm{d}^3\boldsymbol{r}\,.$$

For $I_0 = 0$,

$$(11)\qquad F_2(q) = F_p(q)\,\frac{(4\pi)^{\frac{1}{2}}}{5}\int D_2(r)\,j_2(qr)\,r^2\,\mathrm{d}r\,.$$

We now assume that the deformed nucleus is obtained by performing an incompressible and irrotational deformation on a « nucleus » which was originally spherically symmetric. Then, to first order in the deformation

$$(12)\qquad D_2(r) = \alpha r\,\frac{\mathrm{d}D_0(r)}{\mathrm{d}r}\,,$$

and F_2 is again given by equations (2) and (4) with $L = 2$.

HELM [3] has analysed the elastic scattering of high energy electrons in terms of a folded charge distribution

$$(13)\qquad \varrho_f(r) = \int \varrho_a(r')\,\varrho_b(|\,\boldsymbol{r} - \boldsymbol{r}'\,|)\,\mathrm{d}^3\boldsymbol{r}'\,.$$

Then

$$(14)\qquad F_f(q) = F_a(q)\,F_b(q)\,.$$

[5] B. W. DOWNS, D. G. RAVENHALL and D. R. YENNIE: *Phys. Rev.*, **106**, 1285 (1957).

[6] For notation see A. R. EDMONDS: *Angular Momentum in Quantum Mechanics*. (Princeton, 1957).

ϱ_a is taken to be a step function (uniform charge distribution) and ϱ_b to be a gaussian function.

For inelastic scattering, HELM uses a transition charge density

$$\varrho_t(\boldsymbol{r}) \propto \int \delta(r' - R) P_L(\cos\theta') \varrho_b(|\boldsymbol{r} - \boldsymbol{r}'|)\, \mathrm{d}^3\boldsymbol{r}', \tag{15}$$

and for $E2$ transitions obtains

$$F_2 \propto q F_b(q) \frac{\mathrm{d}}{\mathrm{d}q} F_a(q)\,. \tag{16}$$

Using Helm's folded charge distribution (13) in the incompressible model, and neglecting the proton size, we obtain from equation (6),

$$I_2 = -q \left[F_b(q) \frac{\mathrm{d}}{\mathrm{d}q} F_a(q) + F_a(q) \frac{\mathrm{d}}{\mathrm{d}q} F_b(q) \right]. \tag{17}$$

Comparing this with equation (16) it is seen that Helm's model for inelastic scattering does not correspond to an incompressible nucleus. However, the compression effects are concentrated near the surface of the nucleus.

The corrections for proton size can be easily made if f_p is assumed to be

$$f_p(r) = f_p(0) \exp\left[-r^2/2 g_p^2\right]. \tag{18}$$

Writing

$$\varrho_b(r) = b(0) \exp\left[-r^2/2 g_b^2\right], \tag{19}$$

then

$$D(r) = \int \varrho_a(r') \varrho_a(0) \exp\left[-|\boldsymbol{r} - \boldsymbol{r}'|^2/2 g_d^2\right] \mathrm{d}^3\boldsymbol{r}', \tag{20}$$

where $g_d^2 = g_b^2 - g_p^2$. Then

$$I_2(q) F_p(q) = 3\left[j_2(qR) + (q g_d^2/R) j_1(qR)\right] \exp\left[-q^2 g_b^2/2\right], \tag{21}$$

$$I_3(q) F_p(q) = 3R\left[j_3(qR) + 2(q g_d^2/R) j_2(qR) + (q g_d^2/R)^2 j_1(qR)\right] \exp\left[-q^2 g_b^2/2\right]. \tag{22}$$

Neglecting the proton size, we would have

$$I_2(q) = 3\left[j_2(qR) + (q g_b^2/R) j_1(qR)\right] \exp\left[-q^2 g_b^2/2\right], \tag{23}$$

$$I_3(q) = 3\left[j_3(qR) + 2(q g_b^2/R) j_2(qR) + (q g_b^2/R)^2 j_1(qR)\right] \exp\left[-q^2 g_b^2/2\right]. \tag{24}$$

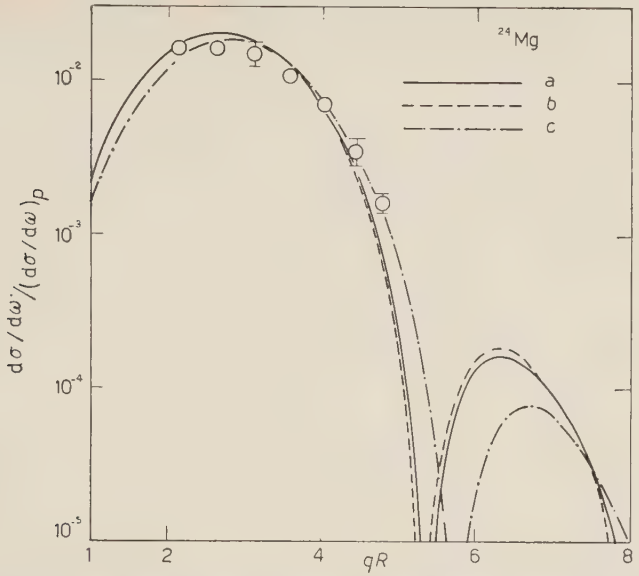

Fig. 1. – Electron excitation of the 1.37 MeV level (2^-) of ^{24}Mg. *a*) incompressible model with correction for finite size of the proton (eq. (21)); *b*) incompressible model with point protons (eq. (23)); *c*) Helm's model (eq. (25)). The experimental points are those of HELM ([3]) for the inelastic scattering of 187 MeV electrons. All theoretical curves may be displaced arbitrarily in the vertical direction.

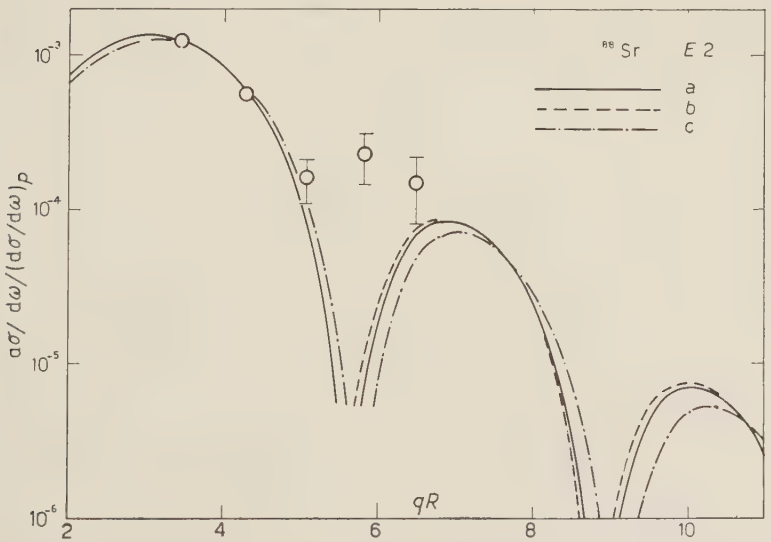

Fig. 2. – Electron excitation of the 1.85 MeV level (2^-) of ^{88}Sr. *a*) incompressible model with correction for finite size of the protons (eq. (21)); *b*) incompressible model with point protons (eq. (23)); *c*) Helm's model (eq. (25)). The experimental points are those of HELM ([3]) for the inelastic scattering of 187 MeV electrons. All theoretical curves may be displaced arbitrarily in the vertical direction.

For comparison, Helm's form factors are

$$F_2(q) \propto j_2(qR) \exp\left[-q^2 g_b^2/2\right], \tag{25}$$

$$F_3(q) \propto j_3(qR) \exp\left[-q^2 g_b^2/2\right]. \tag{26}$$

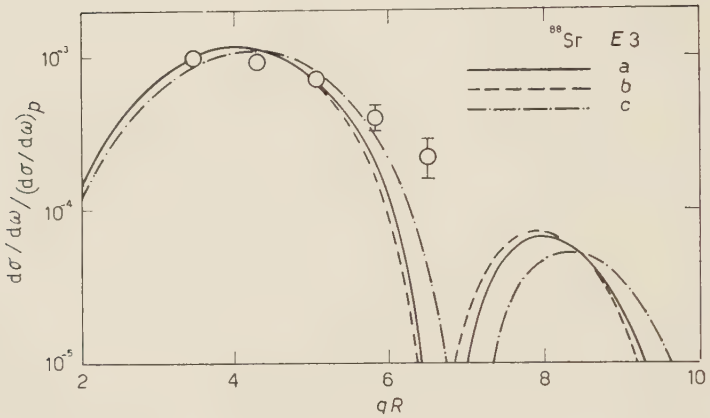

Fig. 3. – Electron excitation of the 2.76 MeV level (3^-) of ^{88}Sr. *a*) incompressible model with correction for finite size of the protons (eq. (22)); *b*) incompressible model with point protons (eq. (24)); *c*) Helm's model (eq. (26)). The experimental points are those of Helm [3] for the inelastic scattering of 187 MeV electrons. All theoretical curves may be displaced arbitrarily in the vertical direction.

Figs. 1, 2 and 3 show the comparison of these form factors with Helm's [3] experimental results for ^{24}Mg and ^{88}Sr. The parameters g_b, R have been taken from Helm's analysis of the elastic scattering, and $0.7 \cdot 10^{-13}$ cm has been used for the mean square radius of the proton. For the range of q of the experimental results the proton size correction is small. The agreement with experiment is slightly better for Helm's form factors than for the incompressible model.

In conclusion, Helm's [3] experimental results seem to provide some indication of nuclear compressibility near the nuclear surface. However, more exact information about nuclear compressibility could be obtained by experiments over a wider range of q, and especially at values of q extending to the second maximum in the form factor.

* * *

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RIASSUNTO (*)

Si ottengono i fattori di forma per l'eccitazione dell'elettrone delle transizioni nucleari collettive vibratorie e rotazionali nell'ipotesi dell'incompressibilità del nucleo, e si confrontano con i risultati teorici e sperimentali di HELM [3]. Lo scattering anelastico di elettroni di alta energia può essere usato per studiare la compressibilità nucleare.

(*) *Traduzione a cura della Redazione.*

On the Longitudinal Depolarization of Beams of Spin $\frac{1}{2}$ Particles by Multiple Coulomb Scattering.

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Summary. — The average longitudinal depolarization of an electron beam by multiple Coulomb scattering has been evaluated. The treatment is based on the iteration, extended to a whole succession of scatterings, of the matrix which describes the variation of the polarization state for a single scattering, and on the average of the iterated matrix on all possible scattering successions. One gets a very simple formula for the average longitudinal polarization degree of the whole outgoing beam, and the final expressions are evaluated by using the second Born approximation. Numerical values of the depolarization for Al and Au targets and for several values of energy and thickness are given; these values in all cases considered are never greater than few percents.

1. – Introduction.

The analysis of the effects of the interactions that polarized electron beams undergo in matter requires, among other things, the knowledge of the depolarization induced in the beam before the interaction takes place.

For example, for the discussion of the results of experiments carried out recently [1] on the polarization of bremsstrahlung produced by electrons in Fe, Ag and Pb target, the knowledge of the order of magnitude of the depolarization of the electrons in the target before the photon emission plays an essential role.

We are considering here only the elastic nuclear coulombian scattering of the electrons.

[1] A. Bisi and L. Zappa: *Phys. Rev. Lett.*, **2**, 384 (1959).

The depolarization of electrons by multiple scattering was first considered in 1939 by BETHE and ROSE (2) with the aim of explaining the lack of asymmetry in the double Coulomb scattering as appeared in some experiments carried out at the time (3). The study of the depolarization by multiples cattering has been recently considered by MÜHLSCHLEGEL and KOPPE (4) and by TOPTYGIN (5), by using very general methods. MÜHLSCHLEGEL and KOPPE use the generalization, given by WALDMANN (6) for the spin, of the Boltzmann kinetic equation, and obtain the angular distribution of the intensity and polarization state of the beam emerging from the target. They consider only small scattering angles and for this reason they neglect the term accounting for the Mott polarization. As concerns this term, this treatment is equivalent to the use of the Born first approximation.

A similar calculation, also based on the kinetic Boltzmann equation, has been made by TOPTYGIN (5). He also gets the angular distribution of the intensity and polarization state of the emerging beam, but he considers also large scattering angles and gives the results in the second Born approximation.

MÜHLSCHLEGEL and KOPPE as well as TOPTYGIN describe the polarization state by using the orientation coefficients (7), or the Stokes parameters (8), widely employed in these last few years for the study of polarization effects.

The Stokes parameters are also used in this paper and the procedure is based on the iteration, extended to a complete succession of scatterings in the target, of the matrix that relates the polarization states after and before the single scattering. One obtains a very simple formula for the average longitudinal polarization degree of the total emerging beam. This polarization degree is expressed in terms of integrals on the Darwin functions $f(\theta)$ and $g(\theta)$. These integrals can be evaluated very easily in the first and second Born approximation, and in these cases the explicit expression of the polarization degree can be obtained. One remarks that the contribution of the second Born approximation is very small: its order of magnitude is not greater than few (1/1 000)'s, in any case here considered.

The validity of the method here used is not, in principle, limited to small scattering angles. On the other hand, as the final result is the longitudinal polarization degree averaged on the total emerging beam, it would be much more significant, if the emerging beam were collimated. This requirement implies that the energy of the incident beam and the thickness of the target

(2) M. E. ROSE and H. A. BETHE: *Phys. Rev.*, **55**, 277 (1939).
(3) E. G. DYMOND: *Proc. Roy. Soc.*, **136**, 638 (1932).
(4) B. MÜHLSCHLEGEL and H. KOPPE: *Zeits. Phys.*, **150**, 474 (1958).
(5) I. N. TOPTYGIN: *Žurn. Èksp. Teor. Fiz.*, **36**, 488 (1959).
(6) L. WALDMANN: *Zeits. Naturfor.*, **12***a*, 660 (1957); **15***a*, 19 (1960).
(7) H. A. TOLHOEK: *Rev. Mod. Phys.*, **28**, 277 (1956).
(8) U. FANO: *Journ. Opt. Soc. Am.*, **39**, 859 (1949).

must have such values as to give prevalently small scattering angles in the material.

We give some numerical results for the depolarization in the case of targets of Al and Au for thicknesses ranging from 10^{-4} to 10^{-2} cm and electron's energies ranging from 10^{-1} to 20 MeV. In all these cases the depolarizations are very small, reaching at the most values of few percents.

2. – Dependence of the polarization on the succession of scattering in the target.

The beam emerging from the target is made up of electrons that have undergone different successions of scatterings in the material. We shall briefly call such a succession a « history ». A history can be represented by a chain of $N+1$ unitary vectors $\boldsymbol{i}_s$ in the space, if N is the total number of collisions that the electron undergoes in the target. We assume that all the histories can be represented by chains of $N+1$ vectors, and for N we shall assume the average number of collisions in the target. With every vector $\boldsymbol{i}_s$ of the chain we associate a rectangular co-ordinate system

$$\boldsymbol{i}_s, \qquad \boldsymbol{j}_s = \frac{(\boldsymbol{i}_{s-1} \wedge \boldsymbol{i}_s) \wedge \boldsymbol{i}_s}{\sin \theta_{s-1}}, \qquad \boldsymbol{k}_s = \frac{\boldsymbol{i}_{s-1} \wedge \boldsymbol{i}_s}{\sin \theta_{s-1}},$$

where θ_{s-1} is the angle between the vectors $\boldsymbol{i}_{s-1}$ and $\boldsymbol{i}_s$, that is the angle of the $(s-1)$th scattering. Therefore the unitary vectors $\boldsymbol{i}_s$, $\boldsymbol{j}_s$ and $\boldsymbol{k}_s$ represent the propagation direction after the $(s-1)$th scattering, the normal to the one in the plane of the $(s-1)$th scattering and the normal to the scattering plane.

The properties of the electrons in the emerging beam are completely determined by the history that they have undergone in the target and therefore they are functions of the vector chain. The geometry of the vector chain is specified by the matrices:

(1) $$A_s = \Theta_s \Phi_s ,$$

(2) $$\Theta_s = \begin{pmatrix} \cos\theta_s & \sin\theta_s & 0 \\ -\sin\theta_s & \cos\theta_s & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(3) $$\Phi_s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_s & \sin\varphi_s \\ 0 & -\sin\varphi_s & \cos\varphi_s \end{pmatrix},$$

which represent the rotations that bring the vectors $\boldsymbol{i}_s$, $\boldsymbol{j}_s$, $\boldsymbol{k}_s$ into $\boldsymbol{i}_{s-1}$, $\boldsymbol{j}_{s-1}$, $\boldsymbol{k}_{s+1}$. φ_s is the angle between the planes $(\boldsymbol{i}_{s-1}, \boldsymbol{i}_s)$ and $(\boldsymbol{i}_s, \boldsymbol{i}_{s-1})$. Of course φ_1 specifies the position of the plane of the first scattering in an arbitrary reference.

It is easy to obtain the polarization state as a function of the history. In order to do this, it is necessary to know the matrix which describes the change of the polarization state for a single Coulomb scattering. Such a matrix has been determined in a previous paper (9), and is expressed in terms of the Darwin functions $f(\theta)$ and $g(\theta)$ as follows:

$$T = (|f(\theta)|^2 + |g(\theta)|^2) \cdot \begin{pmatrix} 1 & 0 & 0 & \nu(\theta) \\ 0 & \lambda(\theta)\cos\theta + \mu(\theta)\sin\theta & \lambda(\theta)\sin\theta - \mu(\theta)\cos\theta & 0 \\ 0 & -\lambda(\theta)\sin\theta + \mu(\theta)\cos\theta & \lambda(\theta)\cos\theta + \mu(\theta)\sin\theta & 0 \\ \nu(\theta) & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

where

$$\begin{cases} \lambda(\theta) = (|f|^2 - |g|^2)/(|f|^2 + |g|^2), \\ \mu(\theta) = 2\,\mathrm{Re}\, fg^*/(|f|^2 + |g|^2), \\ \nu(\theta) = -2\,\mathrm{Im}\, fg^*/(|f|^2 + |g|^2). \end{cases} \tag{5}$$

This matrix relates the Stokes parameters of the incident beam to those of the scattered one:

$$\sum_\nu T_{\mu\nu} P_\nu^{(\mathrm{in})} = P_\mu^{(\mathrm{sc})} \qquad (\mu, \nu = 0 \div 3). \tag{6}$$

If the matrix T is written in the representation (4), P_0 is the beam intensity and P_1, P_2, P_3 describe the polarization with respect to the direction of motion, the normal to the direction of motion in the scattering plane and the normal to this. After introducing the matrix

$$\Phi'_s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \Phi_s & \\ 0 & & & \end{pmatrix} \tag{7}$$

if $P_\nu^{(s)}$ and $P_\mu^{(s+1)}$ $(\mu, \nu = 0 \div 3)$ represent the Stokes parameters of the beam

(9) G. Passatore: *Nuovo Cimento*, **6**, 850 (1957).

before and after the scattering, one has:

$$P_{\mu}^{(s+1)} = \sum_{\nu} (T_s \Phi'_s)_{\mu\nu} P_{\nu}^{(s)}, \tag{8}$$

where $T_s = T(\theta_s)$.

Therefore the polarization state of the electrons that have undergone a certain history is given by:

$$P_{\mu}^{(N+1)} = \sum_{\nu} \prod_{s=1}^{N} (T'_s \Phi'_s)_{\mu\nu} P_{\nu}^{(\mathrm{in})}, \tag{9}$$

where $P_{\nu}^{(\mathrm{in})}$ are the Stokes parameters of the incident beam. In particular, the longitudinal polarization degree $p_l^{(sc)}$ is given, as a function of the history, by:

$$p_l^{(sc)} = \frac{P_1^{(N+1)}}{P_0^{(N+1)}} = \frac{\sum_{\nu} \left(\prod_{s=1}^{N} T_s \Phi'_s \right)_{1\nu} P_{\nu}^{(\mathrm{in})}}{\sum_{\nu} \left(\prod_{s=1}^{N} T_s \Phi'_s \right)_{0\nu} P_{\nu}^{(\mathrm{in})}}. \tag{10}$$

3. – Longitudinal polarization degree of the emerging beam.

We define as longitudinal polarization degree of the total emerging beam the average, taken on all the histories, of the longitudinal polarization degree(10):

$$\overline{p_l^{(sc)}} = \frac{\int p_l^{(sc)} \prod_{s=1}^{N} \sigma_s \, \mathrm{d}\Omega_s}{\int \prod_{s=1}^{N} \sigma_s \, \mathrm{d}\Omega_s}. \tag{11}$$

Here σ_s is the differential cross-section for the s-th scattering. The beam intensity $P_0^{(N+1)}$, the velocity v and the product of the cross-sections are related by:

$$P_0^{(N+1)} = P_0^{(\mathrm{in})} v^N \prod_{s=1}^{N} \sigma_s \tag{12}$$

and the relation (11) becomes, by using (10):

$$\overline{p_l^{(sc)}} = \frac{\int P_1^{(N+1)} \, \mathrm{d}\Omega_N \ldots \mathrm{d}\Omega_1}{\int P_0^{(N+1)} \, \mathrm{d}\Omega_N \ldots \mathrm{d}\Omega_1}. \tag{13}$$

After introducing the matrix

$$\mathscr{T} = \prod_{s=1}^{N} T_s \Phi'_s, \tag{14}$$

one has

$$P_0^{(N+1)} = \sum_{\mu} \mathscr{T}_{0\mu} P_{\mu}^{(\mathrm{in})}, \qquad P_1^{(N+1)} = \sum_{\mu} \mathscr{T}_{1\mu} P_{\mu}^{(\mathrm{in})}. \tag{15}$$

Therefore we are interested in the matrix

$$\int \mathscr{T} \, \mathrm{d}\Omega_N \ldots \mathrm{d}\Omega_1 = \prod_{s=1}^{N} \left(\int T_s \Phi'_s \mathrm{d}\Omega_s \right), \tag{16}$$

where the commutation of the integral with the product is possible because the matrix $T_s \Phi'_s$ depends only on θ_s and φ_s. Furthermore, as T_s depends only on θ_s and Φ'_s only on φ_s, one has:

$$\int T_s \Phi'_s \mathrm{d}\Omega_s = \int_0^{\pi} T(\theta_s) \sin\theta_s \, \mathrm{d}\theta_s \int_0^{2\pi} \Phi'(\varphi_s) \, \mathrm{d}\varphi_s, \tag{17}$$

where

$$\int_0^{2\pi} \Phi'(\varphi_s) \, \mathrm{d}\varphi_s = 2\pi \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{18}$$

Therefore one obtains

$$\int T_s \Phi'_s \mathrm{d}\Omega_s = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & C & 0 & 0 \\ D & 0 & 0 & 0 \end{pmatrix}, \tag{19}$$

where

$$A = 2\pi \int_{\theta_1}^{\theta_2} (|f(\theta)|^2 + |g(\theta)|^2) \sin\theta \, \mathrm{d}\theta, \tag{20a}$$

$$B = 2\pi \int_{\theta_1}^{\theta_2} \{(|f(\theta)|^2 - |g(\theta)|^2) \cos\theta + 2 \operatorname{Re} f(\theta) g^*(\theta) \sin\theta\} \sin\theta \, \mathrm{d}\theta, \tag{20b}$$

$$C = 2\pi \int_{\theta_1}^{\theta_2} \{(|g(\theta)|^2 - |f(\theta)|^2) \sin\theta + 2 \operatorname{Re} f(\theta) g^*(\theta) \cos\theta\} \sin\theta \, \mathrm{d}\theta, \tag{20c}$$

$$D = -4\pi \int_{\theta_1}^{\theta_2} \operatorname{Im} f(\theta) g^*(\theta) \sin\theta \, \mathrm{d}\theta. \tag{20d}$$

θ_1 and θ_2 are the minimum and maximum angles of scattering that will be specified later on.

The reiteration on the matrix (19) is immediate, and gives:

$$\int \mathscr{T}\, \mathrm{d}\Omega_s \dots \mathrm{d}\Omega_\nu = \begin{pmatrix} A^N & 0 & 0 & 0 \\ 0 & B^N & 0 & 0 \\ 0 & B^{N-1}C & 0 & 0 \\ DA^{N-1} & 0 & 0 & 0 \end{pmatrix}. \tag{21}$$

The longitudinal polarization degree of the emerging beam is given, by the relation (13), from:

$$\overline{p_l^{(\mathrm{sc})}} = \frac{\sum_\mu \left(\int \mathscr{T}\, \mathrm{d}\Omega_1 \dots \mathrm{d}\Omega_N\right)_{1\mu} P_\mu^{(\mathrm{in})}}{\sum_\mu \left(\int \mathscr{T}\, \mathrm{d}\Omega_1 \dots \mathrm{d}\Omega_N\right)_{0\mu} P_\mu^{(\mathrm{in})}} = \left(\frac{B}{A}\right)^N p_l^{(\mathrm{in})}, \tag{22}$$

where $p_l^{(\mathrm{in})}$ is the longitudinal polarization degree of the incident beam. The matrix (21) shows that the longitudinal polarization degree of the emerging beam does not depend on the transversal polarization degree of the incident one.

It is easy to verify that the ratio $\overline{p_l^{(\mathrm{sc})}}/p_l^{(\mathrm{in})}$ given by the formula (22) is always less than one. Indeed if $h(\theta) = f(\theta)\,\mathrm{tg}\,(\theta/2)$, the relation $A \geqslant B$ becomes $[\mathrm{Re}\,(h-g)]^2 + \mathrm{Im}\,(h-g)]^2 \geqslant 0$.

It is worth-while observing how the integration over the azimuths φ_s simplifies essentially the problem and allows us to easily relate the depolarization by multiple scattering to the features of the kinematics of the spin in the single scattering, that has been considered by many authors in the last few years ([7,9-12]).

On account of this integration, the problem here considered is completely the same as the other one where the variation of the polarization state at each scattering is described, in place of the matrix $T_s\Phi_s'$, by the matrix:

$$\frac{1}{2\pi} T_s \int_0^{2\pi} \Phi_s' \,\mathrm{d}\varphi_s = (|f|^2 + |g|^2) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda\cos\theta + \mu\sin\theta & 0 & 0 \\ 0 & -\lambda\sin\theta + \mu\cos\theta & 0 & 0 \\ \nu & 0 & 0 & 0 \end{pmatrix}.$$

This means that the polarization state of a beam coming from the s-th scattering is determined only by the longitudinal polarization state of the

([10]) Iu. V. Anishchenko and A. A. Rulchastze: *Žurn. Éksp. Teor. Fiz.*, **33**, 279 (1957).

([11]) F. Gürsey: *Phys. Rev.*, **107**, 1734 (1957).

([12]) M. Bernardini, P. Brovetto and S. Ferroni: *Nucl. Phys.*, **8**, 294 (1958).

incident beam. Moreover the longitudinal polarization degree of the scattered beam depends on the scattering angle θ_s and on the longitudinal polarization degree of the incident beam through the law proper to the Coulomb scattering.

Hence it becomes clear how the term $2\,\mathrm{Im}\,fg^*$ responsible for the spin rotation out of the scattering plane and accounting for the Mott polarization does not contribute to the average longitudinal depolarization of the emerging beam. This gives a first support to the treatment in the first Born approximation, since, as it is known, this term is peculiar to the second and higher approximations because of the invariance of the interaction under time reversal ([13]).

On the other hand, it follows that the depolarization is essentially determined by the way how in a single Coulomb scattering a beam longitudinally polarized changes its longitudinal polarization degree.

As far as concerns the spin kinematics of a longitudinally polarized beam, one easily sees that, if the spin of the incident electron is parallel to the momentum, it conserves approximately its direction for energies small compared to the rest energy, whereas for energies large with respect to the rest energy it tends to rotate with the momentum ([9]). For this reason the longitudinal depolarization of a beam decreases rapidly in passing from low energies, where the momentum undergoes larger changes in direction and the spin remains parallel to itself, to high energies, where small scattering angles predominate and spin and momentum tend to rotate together. Therefore the depolarization is essentially determined by the spreading of the beam. It follows also that, for the same kinetic energy, the spin kinematics is different for the electron and for the μ-meson, because of the different order of magnitude of the mass. For example, for a μ-meson from the π-decay at rest, whose kinetic energy is about 4 MeV, the spin conserves practically its direction ([14]), whereas at this energy for an electron scattered at angles smaller than $\pi/4$ the spin approximately rotates with the momentum. It follows that, for the same kinetic energy, the longitudinal depolarization is greater for a μ-meson than for an electron beam.

While the terms of the first two rows of the matrix (21) are connected with physical properties of the emerging beam, it is not so for the terms of the last two rows. Indeed the quantities:

$$\sum_{\mu}\left(\int \mathscr{T}\,\mathrm{d}\Omega_1 \ldots \mathrm{d}\Omega_N\right)_{2\mu} P_{\mu}^{(\mathrm{in})},$$

and

$$\sum_{\mu}\left(\int \mathscr{T}\,\mathrm{d}\Omega_1 \ldots \mathrm{d}\Omega_N\right)_{3\mu} P_{\mu}^{(\mathrm{in})},$$

([13]) See, for example, Ya. Smorodinskii: *Usp. Fiz. Nauk*, **67**, 43 (1959).
([14]) J. I. Friedman and V. L. Telegdi: *Phys. Rev.*, **106**, 1290 (1957).

cannot be related to properties of the emerging beam as, on the contrary, the quantities $\sum_{\mu}\left(\int \mathscr{T}\, d\Omega_1 \ldots d\Omega_N\right)_{0\mu} P_{\mu}^{(\mathrm{in})}$ and $\sum_{\mu}\left(\int \mathscr{T}\, d\Omega_1 \ldots d\Omega_N\right)_{1\mu} P_{\mu}^{(\mathrm{in})}$ are related to the intensity and to the longitudinal polarization degree.

The reason for the different meaning of the terms of the matrix (21) is that while the intensity and the longitudinal polarization degree as a function of the history have a meaning which does not depend on the history, it is not the same for the transversal polarization degree. This is defined with respect to the plane of the last scattering which in turn is related to the geometry of the whole history. Owing to this, while the average on the histories of the longitudinal polarization degree retains a precise meaning, and can be assumed as the longitudinal polarization degree of the whole emerging beam, this does not happen for the average of the transversal polarization degree, which loses all meaning. Therefore this method applies only to the evaluation of the longitudinal polarization degree. A treatment of the transversal polarization is possible by referring the Stokes parameters of the electrons that have undergone the various histories to a unique system of reference before making the integration.

4. – Evaluation of the depolarization.

For the quantities $|f|^2$, $|g|^2$ and $\mathrm{Re}\, fg^*$ we have assumed their developments up to $(\alpha Z)^3$ [5].

This approximation is equivalent to taking the matrix T in the second Born approximation.

All the integrals are of elementary type, and one obtains:

$$(\overline{p_i^{(\mathrm{sc})}})/p_i^{(\mathrm{in})} = P(W, t, Z)\,, \tag{23}$$

where W is the kinetic energy of the electron in MeV, Z the atomic number of the target element and t the thickness of the target in cm. The function $P(W, t, Z)$ is given by:

$$P(W, t, Z) = a^N b\,, \tag{24}$$

where

$$a = 1 - \frac{2(1-\beta^2)\ln\dfrac{1-\cos\theta_2}{1-\cos\theta_1}}{2\dfrac{\cos\theta_1-\cos\theta_2}{(1-\cos\theta_1)(1-\cos\theta_2)} - \beta^2 \ln\dfrac{1-\cos\theta_2}{1-\cos\theta_1}}\,, \tag{25}$$

$$b = 1 + N\alpha\cdot \frac{4\pi\beta Z(1-\beta^2)\ln\dfrac{1-\cos\theta_2}{1-\cos\theta_1}\left(\mathrm{cosec}\,\dfrac{\theta_1}{2} - \mathrm{cosec}\,\dfrac{\theta_2}{2} - \dfrac{1}{2}\ln\dfrac{1-\cos\theta_2}{1-\cos\theta_1}\right)}{\left(2\dfrac{\cos\theta_1-\cos\theta_2}{(1-\cos\theta_1)(1-\cos\theta_2)} - \beta^2\ln\dfrac{1-\cos\theta_2}{1-\cos\theta_1}\right)\left(2\dfrac{\cos\theta_1-\cos\theta_2}{(1-\cos\theta_1)(1-\cos\theta_2)} + (\beta^2-2)\ln\dfrac{1-\cos\theta_2}{1-\cos\theta_1}\right)} \tag{26}$$

The term on the right hand side of relation (26) containing the fine structure constant is due to the second Born approximation. This contribution is very small: in all the cases considered in the Tables I and II, it does not exceed the order of magnitude of few (1/1000)'s.

TABLE I. – Al *target*.

t_{cm}	W_{MeV}	θ_2	N	d	N' (*)	d' (*)
10^{-4}	10^{-1}	6°	6	5.5 ‰	2	3.0 %
10^{-3}	10^{-1}	16°	72	11.10 %	28	8.86 %
10^{-3}	$5\cdot10^{-1}$	9°	30	2.1 ‰	10	1.5 ‰
10^{-3}	1	2°	25	6 ‱	8	2 ‱
10^{-2}	$5\cdot10^{-1}$	12°	309	3.37 %	111	2.78 %
10^{-2}	1	7°	261	1.08 %	93	3.9 ‰
10^{-2}	2	4°	241	1.2 ‰	86	4 ‱
10^{-1}	2	17°	2423	1.79 %	868	6.4 ‰
10^{-1}	10	3°	2323	<1 ‱	835	<1 ‱
$2\cdot10^{-1}$	10	4°	4660	<1 ‱	1671	<1 ‱

(*) For the two last columns, see the note added in proof.

The function P depends on W, t and Z through the average number of collisions N, and the minimum and maximum scattering angles θ_1 and θ_2.

The minimum scattering angle θ_1 is determined by the screening of the atomic electrons. For this we assume the expression

$$\theta_1 = \left(\frac{Z^{\frac{1}{3}}}{137}\frac{m_0 c}{p}\frac{1}{0.885}\right)\left[1.13 + 3.76\left(\frac{Z}{137\beta}\right)^2\right]^{\frac{1}{2}} \tag{27}$$

already used by MOLIÈRE ([15]). This expression accounts for a correction to the Thomas-Fermi radius and deviations from the Born first approximation for values of $Z/137\beta$ which are not negligible with respect to unity.

([15]) G. MOLIÈRE: *Zeits. Naturfor.*, **3***a*, 78 (1948).

TABLE II. – Au *target*.

t_{cm}	W_{MeV}	θ_2	N	d	N' (*)	d' (*)
10^{-4}	$5\cdot10^{-1}$	8°	14	6.0 $^0/_{00}$	11	5.6 $^0/_{00}$
10^{-4}	1	4°	12	8 $^0/_{000}$	9	7 $^0/_{000}$
10^{-3}	$5\cdot10^{-1}$	25°	146	10.39 $^0/_0$	121	10.02 $^0/_0$
10^{-3}	1	14°	133	1.53 $^0/_0$	99	1.44 $^0/_0$
10^{-3}	2	8°	127	1.8 $^0/_{00}$	91	1.6 $^0/_{00}$
10^{-3}	10	2°	123	<1 $^0/_{000}$	89	<1 $^0/_{000}$
10^{-2}	2	25°	1 280	2.47 $^0/_0$	923	2.36 $^0/_0$
10^{-2}	10	5°	1 238	2 $^0/_{000}$	900	1 $^0/_{000}$
10^{-2}	20	2°	1 234	<1 $^0/_{000}$	900	<1 $^0/_{000}$
10^{-1}	10	18°	12 390	1.9 $^0/_{00}$	9 012	1.4 $^0/_{00}$
10^{-1}	20	9°	12 350	<1 $^0/_{000}$	9 024	<1 $^0/_{000}$
$2\cdot10^{-1}$	20	13°	24 700	<1 $^0/_{000}$	18 050	<1 $^0/_{000}$

(*) For the two last columns, see the note added in proof.

The maximum value θ_2 of the scattering angle has been chosen so that on the average the incident electron will only make one collision in passing through the target of thickness t for which θ exceeds θ_2. Therefore it is given by

$$2\pi nt\int_{\theta_2}^{\pi}\sigma(\theta)\sin\theta\,d\theta = 1\,, \tag{28}$$

where n is the number of atoms for cm^3 and t the thickness in cm. In the case of small angles and in the first Born approximation it results:

$$\theta_2 = \left[t\,\frac{4\pi\, nZ^2r_0^2(1-\beta^2)}{\beta^4}\right]^{\frac{1}{2}}, \tag{29}$$

while in the second Born approximation and without restrictions on the angles θ_2 given by (28) is the solution of the equation:

$$(30) \qquad \frac{\pi n Z^2 r_0^2 (1-\beta^2)}{2\beta^4}\, t \cdot \left[2\,\mathrm{cosec}^2\frac{\theta_2}{2} - 2 + 4\beta^2 \ln\sin\frac{\theta_2}{2} + 4\pi\beta\alpha Z\left(\mathrm{cosec}\,\frac{\theta_2}{2} + \ln\sin\frac{\theta_2}{2} - 1\right)\right] = 1\,.$$

The average number of collisions N which take place in a target of thickness t is given by

$$(31) \qquad N = 2\pi n t \int_{\theta_1}^{\pi} \sigma(\theta)\,\sin\theta\,\mathrm{d}\theta - 1\,.$$

In first Born approximation and for small angles one obtains from (31)

$$(32) \qquad N = 4\pi n t\,\frac{Z^{\frac{4}{3}} r_0^2}{\alpha^2\beta^2}\,,$$

while in the second Born approximation and without restrictions on the angles, N is given by:

$$(33) \qquad N = \frac{\pi n Z^2 r_0^2 (1-\beta^2)}{2\beta^4}\, t \cdot \left[2\,\mathrm{cosec}^2\frac{\theta_1}{2} - 2 + 4\beta^2 \ln\sin\frac{\theta_1}{2} + 4\pi\beta\alpha Z\left(\mathrm{cosec}\,\frac{\theta_1}{2} + \ln\sin\frac{\theta_1}{2} - 1\right)\right] - 1\,.$$

In the Table I and II we report the values of θ and of N, solutions of the equations (30) and (33) and the values of the longitudinal depolarization $d = 1 - \overline{p}_l^{(\mathrm{sc})}/p_l^{(\mathrm{in})}$ for Al and Au targets. These tables refer to electron beams.

We note that in all the cases here considered the depolarization is very small, and it decreases rapidly with the increase of energy.

* * *

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Note added in proof.

Recently NIGAM, SUNDARESAN and WU [16,17] have shown that the correct second Born approximation for the screening angle is given by:

$$\theta_1 = \left[\frac{Z^{\frac{1}{3}}}{137}\frac{mc}{p}\frac{1}{0.885}\mu\right]\left\{1+4Z\alpha\chi_0\left[\left(\frac{1-\beta^2}{\beta}\right)\ln\chi_0+\frac{0.2310}{\beta}+1.4480\beta\right]\right\}^{\frac{1}{2}}, \tag{34}$$

where

$$\chi_0 = \frac{Z^{\frac{1}{3}}}{137}\frac{mc}{p}\frac{1}{0.885}\mu\,, \qquad \mu = 1.80,$$

instead of Molière's relation. By using the relation (34) the average number of collisions N' and the depolarization d' have been calculated. These quantities result somewhat smaller, but in general the orders of magnitude are not changed.

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(16) B. P. NIGAM, M. K. SUNDARESAN and TA-YOU WU: *Phys. Rev.*, **115**, 491 (1959).
(17) M. K. SUNDARESAN: *Phys. Rev.*, **118**, 1072 (1960).

RIASSUNTO

Viene calcolata la depolarizzazione longitudinale media di un fascio di elettroni causata dallo scattering multiplo coulombiano. La trattazione si basa sull'iterazione, estesa a tutta una successione di scatterings, della matrice che descrive la variazione dello stato di polarizzazione causata da uno scattering singolo, e sulla mediazione della matrice iterata su tutte le possibili successioni di scattering. Si ottiene una formula molto semplice per il grado di polarizzazione longitudinale medio del complessivo fascio emergente, e le espressioni finali vengono valutate usando la seconda approssimazione di Born. Si danno valori numerici della depolarizzazone per targhette di Al e Au e per diversi valori dell'energia e dello spessore: questi valori in tutti i casi considerati non superano mai pochi percento.

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Inelastic Scattering of High Energy Photons in the Coulomb Field of a Nucleus with Production of Electron-Positron Pairs (Radiative Pair Production) (*).

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Summary. — We study the radiative pair production and we give the differential cross-section for two kinematical cases. Pair, electron and positron with the same momentum, forward and scattered photon emitted at an angle θ. Photon emitted in the forward direction and pair, still electron and positron with the same momentum, emitted at an angle θ. The calculations are performed with the standard perturbative theory in Born approximation. For small angle θ and for incident γ energy available from modern synchrotrons the two cross-sections are sufficiently high so that they can be measured.

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1. – Introduction.

It is well known that high order electromagnetic processes in which many particles are produced, can be expected in the theory of S-matrix, and so far some calculations have been made using perturbation methods [1]. Some experimental evidence for the existence of electromagnetic multiple processes has been obtained in the last ten years [2] by studying cosmic radiation events. The construction of new high energy machines for electrons and protons should allow a systematic study of these processes in the next few years. However, detailed and exhaustive theoretical calculations of the rate of these processes and of their angular distributions are still missing. This is due essentially because of the great difficulties encountered when evaluating complicated diagrams. Most of the calculations already done consist simply in an estimate of orders of magnitude. By selecting very special kinematical situations in a very few cases, the calculations were made without approximations. In this paper we study the process of radiative production of an electron-positron pair in the Coulomb field of a nucleus which can also be described as a scattering of photons with pair production. Also in this case, because of the algebraic difficulties, we considered special situations. In the laboratory system the following two cases will be treated:

a) Photon emitted in the forward direction and electron and positron emitted at the same angle θ with the same momentum.

b) The pair emitted forward (electron and positron with the same momentum) and the photon scattered at an angle θ. Only the lowest order diagrams contributing to this process were calculated in the Born approximation using Feynmann-Dyson-Wick techniques. We may remark that a point of interest in high energy processes in quantum electrodynamics is the possibility of testing ideas of fundamental theoretical interest, like the possible existence of a fundamental length. This would give a limit to the applicability of the language of local field theories. Many authors have considered electromagnetic processes from this point of view and similar consideration will be developed in the case of radiative production.

(1) J. M. JAUCH and F. ROHRLICH: *The Theory of Photons and Electrons* (1955); V. VOTRUBA: *Phys. Rev.*, **73**, 1468 (1948); A. BORSELLINO: *Helv. Phys. Acta*, **20**, 136 (1947); F. MANDLE and T. H. R. SKYRME: *Proc. Roy. Soc.*, A **215**, 497 (1952); S. N. GUPTA: *Phys. Rev.*, **98**, 1502 (1955); **99**, 1015 (1955); B. DE TOLLIS and R. S. LIOTTA: *Nuovo Cimento*, **5**, 947 (1957).

(2) See for ex. J. E. HOOPER and D. T. KING: *Phil. Mag.*, **41**, 1194 (1950).

2. – The cross-section.

To write down the matrix element contributing to our process at the lowest order we consider the following graph and all others which are obtained by all possible permutations of the photon lines, keeping fixed the electron lines.

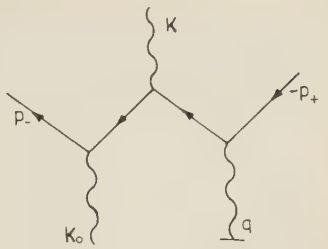

Fig. 1.

The meaning of the symbols in Fig. 1 and in the others mentioned previously is:

$k_0 = (\boldsymbol{k}_0, iE_0)$: 4 momentum of the incoming γ;

$k = (\boldsymbol{k}, iE)$: 4 momentum of the outgoing γ;

$p_- = (\boldsymbol{p}_-, iW_-)$: 4 momentum of the outgoing electron;

$p_+ = (\boldsymbol{p}_+, iW_+)$: 4 momentum of the outgoing positron;

$q = (q_1, q_2, q_3, 0) = k_0 - k - p_- - p_+$: 4 momentum transferred to the nucleus (the kinetic energy of the recoil nucleus has been neglected);

$A_0 = (\boldsymbol{A}_0, 0)$: 4 polarization vector of the incoming γ satisfying $(\boldsymbol{k}_0 \boldsymbol{A}_0) = 0$;

$A = (\boldsymbol{A}, 0)$: 4 polarization vector of the outgoing γ satisfying $(\boldsymbol{k}\boldsymbol{A}) = 0$.

We shall write the external Coulomb field of the nucleus

$$A_\mu^{(e)}(x) = \frac{1}{(2\pi)^4} \int \Phi(q) a_\mu \exp[iqx] \, \mathrm{d}_4 q \,,$$

where

$$\Phi(q) = 2\pi \, \delta(q_0) \frac{Ze}{q^2} \qquad \text{and} \qquad a_\mu \equiv (0, 0, 0, i)$$

($q \cdot x$ is the scalar product of the 4-vectors q and x).

In this notation the matrix element for this process can be written ($\hbar = c = 1$)

$$(1) \qquad M = 2\pi \, \delta(q_0) \frac{Ze^4}{q^2} \bar{u}_- \left\{ \tilde{a} \frac{\tilde{p}_- + \tilde{q} + im}{(p_- + q)^2 + m^2} \tilde{A}_0 \frac{-\tilde{p}_+ - \tilde{k} + im}{(p_+ + k)^2 + m^2} \tilde{A} + \right.$$

$$- \tilde{a} \frac{\tilde{p}_- - \tilde{q} + im}{(p_- + q)^2 + m^2} \tilde{A} \frac{-\tilde{p}_+ + \tilde{k}_0 + im}{(p_+ - k_0)^2 + m^2} \tilde{A}_0 + \tilde{A}_0 \frac{\tilde{p}_- - \tilde{k}_0 + im}{(p_- - k_0)^2 + m^2} \tilde{a} \frac{-\tilde{p}_+ - \tilde{k} - im}{(p_+ + k)^2 + m^2} \tilde{A} +$$

$$+ \tilde{A} \frac{\tilde{p}_- + \tilde{k} + im}{(p_- + k)^2 + m^2} \tilde{a} \frac{-\tilde{p}_+ + \tilde{k}_0 + im}{(p_+ - k_0)^2 + m^2} \tilde{A}_0 + \tilde{A}_0 \frac{\tilde{p}_- - \tilde{k}_0 + im}{(p_- - k_0)^2 + m^2} \tilde{A} \frac{-\tilde{p}_+ - \tilde{q} + im}{(p_+ + q)^2 + m^2} \tilde{a} +$$

$$\left. + \tilde{A} \frac{\tilde{p}_- + \tilde{k} + im}{(p_- + k)^2 + m^2} \tilde{A}_0 \frac{-\tilde{p}_+ - \tilde{q} + im}{(p_+ + q)^2 + m^2} \tilde{a} \right\} u_+ ,$$

where u_- and u_+ are the spinors describing the electron and the positron, m is the electron mass and $\tilde{k}$ means $\gamma_\mu k_\mu$. As already mentioned we have considered the special situation

$$p_- = p_+ = p \,. \tag{2}$$

The cross-section can then be written

$$\mathrm{d}\sigma = r_0^2 \frac{Z^2\alpha^2}{(2\pi)^4} \frac{1}{2} \sum |\bar{u}_- O u_+|^2 \frac{\boldsymbol{P}^2 E}{q^4 E_0} \mathrm{d}W_+ \mathrm{d}W_- \mathrm{d}\Omega_+ \mathrm{d}\Omega_- \mathrm{d}\Omega_\gamma \,, \tag{3}$$

where $\frac{1}{2}\sum$ means the sum over electron and positron spin, the sum over outgoing γ polarization and the average of the incoming γ polarization. O is the operator given explicitly in formula (1). We discuss now in detail the two kinematical situations described in the introduction under *a*) and *b*), in which we may expect the cross-section to be particularly important.

3. – The photon emitted forward and the pair at an angle θ.

Carrying out the operations indicated in formula (3) we obtain in the laboratory system

$$\mathrm{d}\sigma = \frac{1}{2} r_0^2 \frac{Z^2\alpha^2}{(2\pi)^4} \frac{\boldsymbol{P}^2 E}{q^4 W^2 E_0} \mathrm{d}W_+ \mathrm{d}W_- \mathrm{d}\Omega_+ \mathrm{d}\Omega_- \mathrm{d}\Omega_\gamma \cdot \tag{4}$$

$$\cdot \left\{ \frac{11W^2 - 7}{(W - p\cos\theta)^2} - \frac{4p\cos\theta}{(W - p\cos\theta)^3} + \frac{p^2\sin^2\theta[16W^4 + E_0^2E^2 + 10E_0EW(W - p\cos\theta)]}{E_0^2E^2(W - p\cos\theta)^4} \right\} .$$

All energies and momenta are expressed in electron mass units. In Fig. 2 we give the cross-sections corresponding to the following choices of the energies (for $Z = 1$):

(I) $\quad E_0 = 2\cdot 10^3$ e.m. ; $\quad E = \frac{1}{2}E_0$; $\quad W = \frac{1}{4}E_0$;

(II) $\quad E_0 = 2\cdot 10^3$ e.m. ; $\quad E = 1.96\cdot 10^3$ e.m. ; $\quad W = 20$ e.m.

The energy of the incoming γ has been chosen in relation to the energy obtainable with the modern synchrotrons. We should remark that a condition to be satisfied in order that formula (4) be valid at all angles is

$$q \ll M \,,$$

where: $q^2 = 8W^2(1 - \beta\cos\theta) - 4m^2$; $\quad \beta$ = velocity of the electron;

M = mass of the nucleus; $\quad m$ = electron mass.

Otherwise the recoil of the nucleus should be taken into account. The cross-section is exceptionally high at 0°. However if we consider that any measurement represents an average over a small angle, the actual forward

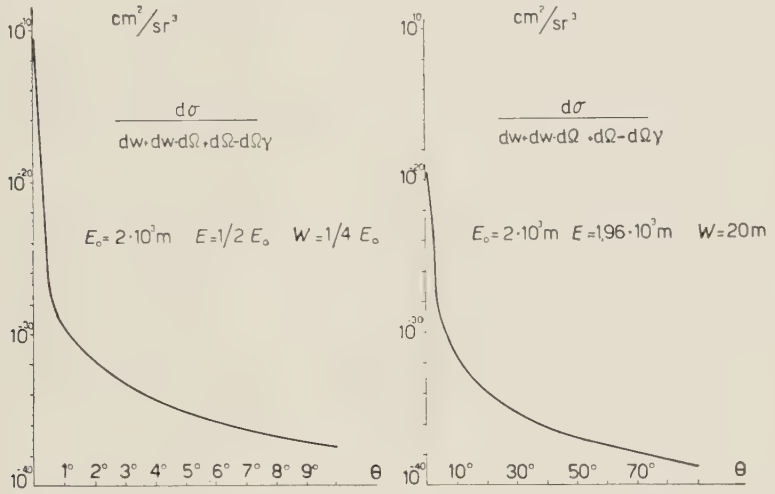

Fig. 2. – Photon emitted forward: angular distribution of the pair.

cross-section is considerably lower. If, for example, we integrate over an angle of the order of one degree around the forward direction, we obtain the reasonable value of 10^{-26} cm²/(sr)³. We point out the strong dependence on the pair energy W. In the two cases discussed the angular dependence is very different and in the second situation the cross-section at 90° is of the same order as the cross-section in the first case at 10°.

At this point we add some consideration on the possibility of using this process to test quantum electrodynamics at small distances. From the above discussion it is apparent that a sufficiently low value of W should be chosen in order to have an observable cross-section at large angles. From an approximate calculation using a simple cut-off model it is easily seen that if the electron propagator had to be modified [for example by substituting $1/(p^2+m^2)$ with $(1/(p^2+m^2))(\Lambda^2/(p^2+m^2+\Lambda^2))$, where p is the momentum of the electron line] the cross-section would be multiplied by a factor $1-2((p^2+m^2)/\Lambda^2)$. In this case using the data corresponding to the second group of values ($W=20$ e.m.) for $\theta=40°$ one could explore distances of the order of 0.2 fermi, by observing a variation in the cross-section of 20%. We remark that at 40° the cross-section could still be observed. In this kind of experiment the target nucleus should be hydrogen since the form factor is not well known for other nuclei. The approximation made neglecting the recoil of the nucleus is valid

since in this case (pair emitted at 40°) the momentum transfer is of the order of 60 MeV and then still $\ll M$. The above discussion is, of course, only suggestive. The problem should be studied more carefully in an experiment aimed at testing quantum electrodynamics at small distances ([3]).

4. – Pair emitted forward and photon scattered at an angle θ.

The cross-section for this case is more complicated and can be written in compact form

$$(5) \qquad d\sigma = \frac{1}{2} r_0^2 \frac{Z^2\alpha^2}{(2\pi)^4} \frac{\boldsymbol{p}^2 E}{q^4 E_0} dW_+ dW_- d\Omega_+ d\Omega_- d\Omega_\gamma \cdot \frac{P_0 + P_1\chi + P_2\chi^2 + P_3\chi^3 + P_4\chi^4 + P_5\chi^5 + P_6\chi^6}{2^8 E_0^2 E^2 W^4 (1-\beta^2)(1-\beta\chi)^2 [W(W-p) + \frac{1}{2}E(E_0-p)(1-\chi)]^2},$$

where $\chi = \cos\theta$ and the P_i are polynomials in $E_0^x E^y p^z$ which we list in the tables below ([4]). All the other symbols are known. In Fig. 3 we give the angular distribution for

$$E_0 = 2\cdot 10^3\, m_e, \qquad E = \tfrac{1}{2}E_0, \qquad W = \tfrac{1}{4}E_0,$$

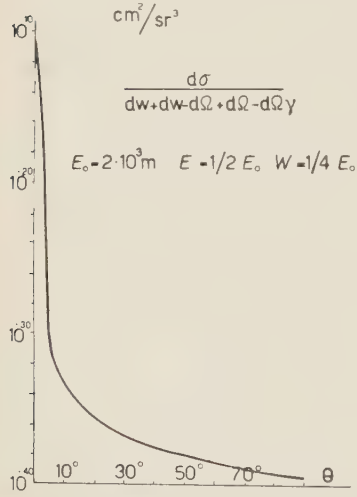

Fig. 3. – Pair emitted forward: angular distribution of the photon.

The situation is not very different from the case of the photon emitted forward and similar considerations are valid in this case. This case however, is interesting since it allows a comparison of this more complicated process with ordinary pair production. The ordinary pair production is exactly 0 in the forward direction; while if we integrate our cross-section over the angles of the emitted photon a rather large cross-section can still be expected since the angular distribution is strongly peaked forward. If the photon is not observed the radiative production would appear as an ordinary pair production. The

([3]) S. D. DRELL: *Ann. Phys.*, **4**, 75 (1958); J. D. BJORKEN, S. D. DRELL and S. C. FRAUTSCHI: *Phys. Rev.*, **112**, 1409 (1958); J. D. BJORKEN and S. D. DRELL: *Phys. Rev.*, **114**, 1368 (1959).

([4]) The tables mean

$$P_0 = 2(-22E_0^7E^2p + 60E_0^6E^3p - 104E_0^5E^4p + \ldots) + 8(12E_0^6E^2 - 22E_0^5E^3 + 44E_0^4E^4 - \ldots)m^2.$$

cross-section (5) is valid if $q \ll M$, where, now,

$$q^2 = 4E_0W(1-\beta) + 2E_0E(1-\cos\theta) - 4EW(1-\beta\cos\theta) - 4m^2 .$$

5. – Validity of the Born approximation and screen effect due to the atomic electrons.

In this section we want to discuss the limits in extending formulae (4) and (5) to any target nucleus (5). For definiteness we shall refer to the case of the photon emitted in the forward direction and then to formula (4), but all the following remarks apply to formula (5) with only minor modifications.

As far as the validity of the Born approximation is concerned it is well known that the following condition has to be satisfied

$$\frac{2\pi Z\alpha}{\beta} \ll 1 ,$$

where $\alpha = 1/137$ and β is the velocity of the electron or of the positron. For values of W not exceedingly low, β is of the order unity and then the theory is valid for nuclei with $Z < 20$.

To discuss the effect of the screening of atomic electrons we estimate the order of magnitude of the momentum transfer q. In the case of the photon emitted in the forward direction we have

$$q^2 = 8W^2(1-\beta\cos\theta) - 4 \quad \text{(e.m.)}^2 .$$

It is evident that with W fixed (and then with fixed β), we shall find $q_{\min}$ for $\theta = 0°$, in which case

$$q_{\min} \approx 1/W$$

and the impact parameter $r_{\max} = 1/q_{\min}$ will increase as W. For $W \approx 10^3$ e.m. we have $r_{\max}$ of the order of 10^{-8} cm. The screening will not be effective if the impact parameter is not greater than the atomic radius a which is of the order 10^{-8} cm. More precisely, according to the Thomas-Fermi model, $a = Z^{-\frac{1}{3}} 0.5 \cdot 10^{-8}$ cm. So we can conclude that, provided we do not consider nuclei with very high Z, (in which case the Born approximation too is not valid), we can go up to energies of the order (250÷300) MeV for W. Further-

(5) W. Heitler: *The Quantum Theory of Radiation* (1954); H. A. Bethe and L. C. Maximon: *Phys. Rev.*, **93**, 768 (1954).

TABLE I.

$E_0^x E^y p^z$																										
x	8	7	6	5	4	3	2	7	6	5	4	3	2	6	5	4	3	2	5	4	3	2	4	3	2
y	2	3	4	5	6	7	8	2	3	4	5	6	7	2	3	4	5	6	2	3	4	5	2	3	4
z								1	1	1	1	1	1	2	2	2	2	2	3	3	3	3	4	4	4
$2\cos^0\theta$								22	60	−104	108	− 66	24	136	−248	296	− 168	48	−280	256	−168		192		
$2\cos\theta$	10	−40	86	−112	86	− 40	10	− 86	300	−472	428	−210	40	268	− 788	844	−524	136	−360	864	−504	192	176	− 336	
$2\cos^2\theta$	6	−46	106	−132	106	− 46	6	42	312	−612	584	−306	64	116	−796	1 204	−740	184	−152	880	−888	256	80	− 336	192
$2\cos^3\theta$		−12	48	− 72	48	−12		− 2	96	−300	344	−162	24	4	−276	676	−540	136	8	368	−632	256	− 16	− 208	160
$2\cos^4\theta$	6	−22	50	− 68	50	−22	6	−24	68	−140	140	− 44		4	− 12	108	−100		80	−80	− 96	96	− 80	16	96
$2\cos^5\theta$		−12	32	− 40	32	−12			44	−108	108	− 68	24		8	48	8			−176	80	− 96		160	
$2\cos^6\theta$			8	− 16	8					− 24	48	− 24				− 8	− 48	24			96				− 96

TABLE II.

$E_0^x E^y p^z$																								
x	6	5	4	3	2	5	4	3	2	6	5	4	3	2	5	4	3	2	4	3	2
y	2	3	4	5	6	2	3	4	5		1	2	3	4		1	2	3		1	2
z						1	1	1	1	2	2	2	2	2	3	3	3	3	4	4	4
$8m^2\cos^0\theta$	12	−22	44	−22	12	−50	62	−62	10	8	− 8	− 8	36	4	−32	16	72	− 64	32		−64
$8m^2\cos\theta$	−10	8	−44	8	−10	46	− 60	94			−16	16	80	− 96		48	40	− 80		32	
$8m^2\cos^2\theta$	−10	36	−44	36	−10	68	−162	144	−74	− 8	8	−108	128	−76	32	−16	16	24	− 32		16
$8m^2\cos^3\theta$		12	8	12		14	− 56	22	−12		16	−136	168	−32		−48	40	−112		32	
$8m^2\cos^4\theta$	− 6	10	−32	10	− 6	18	− 32	50	4			− 4	44	−24			88	− 40			−48
$8m^2\cos^5\theta$		12	− 8	12			− 40	16	−24				24	8				16			

more, if the screening effect is present it would modify the cross-section for small angles while the non-validity of the Born approximation would give corrections at large angles. In this case our cross-sections are so low that no observable effect would be seen.

* * *

The authors wish to thank Professor BRUNO FERRETTI for having suggested the problem and the Istituto Nazionale per le Applicazioni del Calcolo for having performed a part of the algebraic calculations.

RIASSUNTO

Si studia la produzione radiativa di coppie e viene data la sezione d'urto per due casi cinematici. Coppia (elettrone e positrone con lo stesso impulso) in avanti e fotone smesso ad un angolo θ. Fotone in avanti e coppia (con elettrone e positrone aventi ancora lo stesso impulso) emessa ad un angolo θ. Per lo sviluppo dei calcoli si è usata la consueta teoria delle perturbazioni nell'approssimazione di Born. Per piccoli angoli θ e per energie del γ incidente raggiungibili con i moderni sincrotroni le sezioni d'urto sono abbastanza alte da essere misurabili.

Struttura molecolare e teoria statistica dei momenti elettrici di polimeri lineari.

I – Studio del glicole poliossietilenico (*).

V. Magnasco

Istituto di Chimica Industriale dell'Università - Genova

(ricevuto il 23 Luglio 1960)

Riassunto. — Il valor medio quadratico del momento elettrico per il glicole poliossietilenico è stato calcolato col metodo delle matrici di rotazione per tre differenti strutture molecolari, ammettendo la flessibilità interna della macromolecola in soluzione e l'indipendenza statistica delle rotazioni. I tre modelli sono: 1) catena con rotazioni interne statisticamente uniformi; 2) catena periodica con periodo tre; 3) catena uniforme con rotazioni differenti alle due estremità. Le rotazioni interne, che possono anche essere asimmetriche rispetto alla posizione « trans », sono impedite da una barriera periodica di potenziale del tipo $U(\varphi)=(U_0/2)[1-\cos(\varphi-\varphi_0)]$. Formule esplicite sono date per i tre modelli molecolari nel caso di un numero x di periodi della catena abbastanza grande. Per qualunque valore di x è possibile il calcolo numerico di $\overline{\mu^2}$ ricorrendo alle formule di rappresentazione di Lagrange.

1. – Introduzione.

Anche se esistono alcune incertezze nella estrapolazione dei dati di polarizzazione molecolare del soluto a diluizione infinita negli ultimi anni si sono venuti accumulando dei dati sperimentali abbastanza precisi sulle proprietà dielettriche di polimeri in soluzione di solventi poco o non polari [1-6].

(*) Lavoro eseguito con il contributo del C.N.R.

(1) J. Marchal e H. Benoit: *Journ. Chim. Phys.*, **52**, 818 (1955).

(2) T. Uchida, Y. Kurita e M. Kubo: *Journ. Polymer Sci.*, **19**, 365 (1956).

(3) T. Uchida, Y. Kurita, N. Koizumi e M. Kubo: *Journ. Polymer. Sci.*, **21**, 313 (1956).

(4) L. De Brouckère, D. Buess e L. K. H. Van Beek: *Journ Polymer Sci.*, **23**, 233 (1957).

(5) W. R. Krigbaum e A. Roig: *Journ. Chem. Phys.*, **31**, 544 (1959).

(6) R. Bacskai e H. A. Pohl: *Journ. Polymer Sci.*, **42**, 151 (1960).

La variazione del momento elettrico con il peso molecolare e la temperatura può fornire utili informazioni sulla struttura di macromolecole polari e sulle barriere di potenziale che si oppongono alla libera rotazione di una parte della molecola rispetto all'altra.

Un utile contributo può anche derivare dallo studio dell'effetto del solvente sul momento elettrico apparente dei polimero in soluzione, come è stato suggerito da MANDEL ([7]) e DE BROUCKÈRE ([8]).

Se si prescinde da questi ultimi effetti, lo studio teorico del momento elettrico di un polimero è strettamente connesso con quello delle configurazioni. Per effetto dell'agitazione browniana la macromolecola cambia continuamente la sua configurazione microscopica, e il momento elettrico totale risulta da una media su tutti i possibili stati di rotazione interna della molecola, la cui probabilità di occorrenza è intimamente legata alle barriere di potenziale.

È nostra intenzione studiare teoricamente col metodo delle matrici di rotazione ([9,10]) i momenti elettrici di alcune macromolecole caratteristiche che presentino un certo grado di flessibilità locale in relazione alla loro struttura in soluzione.

In questa nota ci occupiamo della teoria relativa al glicole poliossietilenico ([1,3]), mentre rimandiamo alla parte II la discussione e i calcoli dettagliati che sono in corso presso il Centro Calcolo dell'Università di Genova.

2. – Il momento elettrico di una configurazione microscopica.

Consideriamo la catena schematizzata nella Fig. 1. È costituita dalla successione di $N+1$ atomi che immaginiamo localizzati nei punti P_0, P_1, ..., P_{N+1} uniti tra loro da N legami orientati.

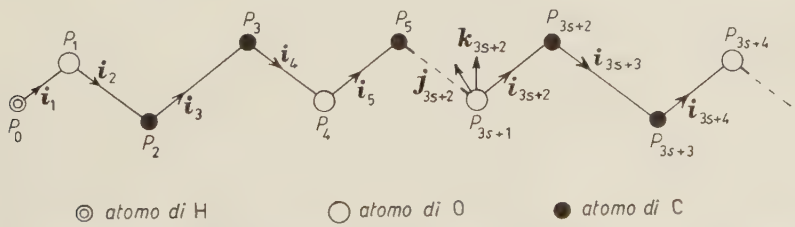

Fig. 1. – Rappresentazione schematica della catena di poliossietilenglicole nella configurazione di massima estensione. Non sono indicati per semplicità gli atomi di idrogeno.

([7]) M. MANDEL: *Journ. Polymer Sci.*, **23**, 241 (1957).

([8]) L. DE BROUCKÈRE e M. MANDEL: *Advances in Chemical Physics*, vol. **1** (New York, 1958), p. 77.

([9]) S. LIFSON: *Journ. Chem. Phys.*, **29**, 80, 89 (1958).

([10]) A. BORSELLINO: *Ric. Scient.*, **29**, 496 (1959); *Journ. Chem. Phys.*, **30**, 857 (1959).

Definiamo x come il numero di unità strutturali identiche che si susseguono nella catena, allora $N = 3x + 2$.

Se supponiamo di trascurare il piccolo contributo derivante dal momento elettrico risultante del gruppo CH_2, possiamo considerare il momento totale della molecola come dovuto alla somma vettoriale dei momenti elementari H—O e C—O, giacenti nella direzione dei rispettivi legami.

Sia μ_A il momento elementare del gruppo alcoolico, μ_E il momento elementare del gruppo etereo.

Associamo a ogni legame una terna destrorsa di versori ortogonali $\boldsymbol{i}_s, \boldsymbol{j}_s, \boldsymbol{k}_s$ prendendo $\boldsymbol{i}_s$ nella direzione del legame s, $\boldsymbol{j}_s$ nel piano individuato da $\boldsymbol{i}_s$ e da $\boldsymbol{i}_{s-1}$, $\boldsymbol{k}_s$ normale al piano dei primi due e orientato nel senso del vettorprodotto $\boldsymbol{i}_s \times \boldsymbol{j}_s$: Avremo allora, riferendoci per fissare le idee all's-esimo momento elementare etereo,

$$\boldsymbol{\mu}_s = \boldsymbol{i}_s \mu_E = (\mu_E, 0, 0) \,. \tag{1}$$

Sia θ il supplemento dell'angolo di valenza, che assumiamo eguale per $\widehat{HOC}$, $\widehat{OCC}$, $\widehat{COC}$ [11]. Per effetto del moto browniano ogni legame può rotare attorno al precedente, e questo comporta la rotazione dei momenti elementari associati con questi legami. Lo stato di rotazione interna del momento $\boldsymbol{\mu}_{s+1}$ è individuato dall'angolo diedro φ_s, contato a partire dalla configurazione « trans » e positivo se la rotazione è antioraria [9,10,12].

Il momento elettrico totale della molecola, nello stato individuato dagli angoli di rotazione interna $\varphi_1, \varphi_2, \ldots, \varphi_{3x+1}$ è dato da

$$\boldsymbol{\mu} = \mu_A(\boldsymbol{i}_1 - \boldsymbol{i}_{3x+2}) + \mu_E \sum_{s=1}^{x} (\boldsymbol{i}_{3s+1} - \boldsymbol{i}_{3s-1}) \,, \tag{2}$$

se facciamo la convenzione di prendere come verso positivo del momento elementare quello diretto dall'atomo a carattere positivo verso quello a carattere negativo.

Introduciamo ora la matrice di rotazione ortogonale A_s che porta la terna s-esima a coincidere con la terna $(s+1)$-esima

$$A_s = \begin{bmatrix} \alpha & -\beta \cos \varphi_s & \beta \operatorname{sen} \varphi_s \\ -\beta & -\alpha \cos \varphi_s & \alpha \operatorname{sen} \varphi_s \\ 0 & -\operatorname{sen} \varphi_s & -\cos \varphi_s \end{bmatrix} \tag{3}$$

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[11] Vedere ad esempio *Tables of Interatomic Distances and Configuration in Molecules and Ions*, (London, 1958).

[12] V. Magnasco: *Ric. Scient.*, **30**, 405 (1960).

dove abbiamo posto, nell'ipotesi di un angolo di valenza costante

$$(4) \qquad \begin{cases} \alpha = \cos\theta\,, \qquad \beta = \operatorname{sen}\theta \\ \alpha^2 + \beta^2 = 1\,. \end{cases}$$

Possiamo scrivere l'equazione (2) nella corrispondente notazione matriciale

$$(5) \qquad \boldsymbol{\mu} = (\mu_A, 0, 0)(I - A_{3x+2,1}) + (\mu_E, 0, 0)\sum_{s=1}^{x}(A_{3s+1,1} - A_{3s-1,1})$$

con le convenzioni già adottate in precedenti lavori ([10,12])

$$(6) \qquad A_{ts} = A_{t-1}A_{t-2}\ldots A_{s+1}A_s$$

$$(7) \qquad A_{ss} = I$$

$$(8) \qquad A_s A_s^* = A_s A_s^{-1} = I$$

dove I è la matrice identità del terzo ordine che trasforma la terna s-esima in se stessa, e A_s^* la matrice transposta di A_s.

Il quadrato del momento elettrico totale della molecola in una data configurazione microscopica è allora espresso dalla relazione

$$(9) \qquad \mu^2 = [(\mu_A, 0, 0)(I - A_{3x+2,1}) + (\mu_E, 0, 0)\sum_{s=1}^{x}(A_{3s+1,1} - A_{3s-1,1})]\cdot$$

$$\cdot[(I - A^*_{1,3x+2})(\mu_A, 0, 0)^* + \sum_{t=1}^{x}(A^*_{1,3t+1} - A^*_{1,3t-1})(\mu_E, 0, 0)^*]\,.$$

3. – Barriere di potenziale e struttura del poliossietilenglicole.

Nella molecola di poliossietilenglicole ci sono da aspettarsi teoricamente barriere di potenziale derivanti da diversi contributi. I legami σ tra carbonio-carbonio e carbonio-ossigeno derivanti dalle ibridizzazioni sp degli atomi di carbonio e di ossigeno, hanno una completa simmetria assiale che dovrebbe, in prima approssimazione, permettere la libera rotazione di una parte della molecola rispetto all'altra.

Ma già da un punto di vista quantomeccanico dovremo aspettarci dei contributi repulsivi dalle interazioni coulombiane e di scambio tra orbitali non legati e dalle interazioni coulombiane tra i nuclei di ossigeno, solo in parte bilanciate dalle attrazioni coulombiane elettroni-nuclei, secondo la for-

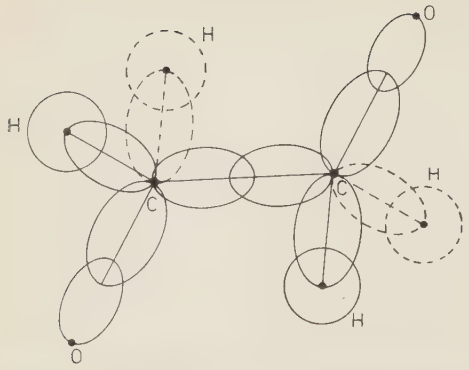

Fig. 2. - Rappresentazione schematica degli orbitali atomici ibridi di C e O e degli orbitali 1s di H che determinano i legami σ O—C, C—C, C—H nell'elemento strutturale O-$(CH_2)_2$-O del glicole poliossietilenico. Gli orbitali tratteggiati sono situati al di là del piano del foglio.

mula del perfetto accoppiamento per le molecole poliatomiche [13]

$$(10) \quad E' = \frac{1}{R_{O,O}} + \sum_{i,j} Q_{ij} - \frac{1}{2} \sum_{i,j} \alpha_{ij}$$

dove E' rappresenta quella parte dell'energia di legame totale che dipende dall'angolo di rotazione interna φ del gruppo O—CH_2 rispetto all'altro gruppo CH_2—O. Abbiamo indicato con $R_{O,O}$ la distanza tra i nuclei di ossigeno, con Q_{ij} e α_{ij} rispettivamente le energie coulombiane e di scambio tra le coppie di orbitali i, j, la doppia somma essendo estesa a tutte le coppie possibili di orbitali non legati i, j. Le energie coulombiane e di scambio sono funzioni periodiche dell'angolo φ, e il valore minimo di E' rispetto a φ determina la configurazione di equilibrio dell'elemento strutturale O—$(CH_2)_2$—O. La valutazione quantitativa di questi effetti non è semplice soprattutto per le notevoli incertezze nel valore numerico degli integrali.

In quanto abbiamo finora detto si trascura l'effetto della polarità dei legami, che è invece notevole nel legame C—O e che determina una distribuzione asimmetrica di carica attorno ai due nuclei.

Questo effetto può essere semplicemente discusso, in una prima approssimazione, dal punto di vista classico considerando le interazioni dipolo-dipolo ed eventualmente le interazioni di ordine superiore [14,15].

Queste considerazioni mostrano come ci sia in realtà da aspettarsi una barriera di potenziale che si oppone alla libera rotazione dei legami, sebbene non sia facile calcolare teoricamente il valore della barriera, inteso come differenza tra l'energia massima e quella che compete alla configurazione di equilibrio.

Informazioni abbastanza precise si possono ottenere sperimentalmente dall'osservazione degli spettri di assorbimento nell'infrarosso e dagli spettri Raman. Partendo dalla considerazione che le vibrazioni « rocking » del gruppo

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[13] C. A. Coulson: *Valence* (Oxford, 1953).

[14] C. P. Smith, R. W. Dornte e E. B. Wilson Jr.: *Journ. Am. Chem. Soc.*, **53**, 4242 (1931).

[15] F. N. Lassettre e L. B. Dean: *Journ. Chem. Phys.*, **17**, 317 (1949).

CH_2 sono molto sensibili alla configurazione molecolare, diversi Autori [16-18] hanno studiato l'assorbimento infrarosso del glicole polietilenico in vari stati di aggregazione nella regione (700÷1 500) cm^{-1}. C'è qualche discordanza nell'assegnazione delle frequenze caratteristiche, che è dovuta principalmente alla differenza nel peso molecolare dei campioni studiati. Le frequenze sono assegnate assumendo che nell'elemento strutturale caratteristico del glicole polietilenico si abbiano solo piccoli spostamenti rispetto alle frequenze della molecola più semplice di 1,2-dicloroetano il cui dettagliato studio teorico è stato fatto da NAKAGAWA e MIZUSHIMA [19].

DAVISON [16] che ha studiato un campione di peso molecolare relativamente elevato ($M = 6\,000$) allo stato solido, attribuisce i forti assorbimenti osservati a 844 e 947 cm^{-1} alle vibrazioni « rocking » del gruppo CH_2 nella configurazione « gauche ». Da ciò egli deduce una struttura elicoidale fortemente compatta, a simmetria ternaria, determinata dalla successione di unità strutturali « gauche » in accordo con i risultati di diffrazione coi raggi X di SAUTER [20]. La cella elementare contiene 4 catene di 9 unità monomeriche ciascuna dentro un periodo di 19.5 Å.

KURODA e KUBO [17] hanno esaminato in un primo tempo i termini della serie a peso molecolare molto basso, dal monomero all'eptamero allo stato liquido puro, mentre in un secondo lavoro hanno esteso l'osservazione infrarossa a termini con grado di polimerizzazione di 14, 18, 23, 30, 34 e 90 allo stato liquido puro e in soluzione benzenica. Lo spettro infrarosso osservato allo stato liquido puro è molto simile a quello osservato in soluzione.

Le loro conclusioni si basano sulla graduale scomparsa della banda a 885 cm^{-1} col crescere del peso molecolare, che essi interpretano come dovuta a una diminuita importanza della vibrazione « rocking » del CH_2 nella sua forma « trans ».

Un fenomeno analogo è stato quasi contemporaneamente riscontrato da WHITE e LOVELL [18] che attribuiscono l'assorbimento a circa 880 cm^{-1} alla configurazione « trans » del gruppo terminale —CH_2—CH_2OH.

La diminuzione nell'intensità di questa banda è spiegata con la diminuzione percentuale dei gruppi terminali coll'aumentare del peso molecolare. Si avrebbe quindi una transizione continua da una struttura in cui è predominante la configurazione « trans » per i bassi pesi molecolari, alla struttura « gauche » riscontrata per i pesi molecolari più elevati. La struttura « gauche » può essere inoltre stabilizzata dalla formazione di legami di idrogeno intramolecolari.

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[16] W. H. T. DAVISON: *Journ. Chem. Soc.*, 3270 (1955).
[17] Y. KURODA e M. KUBO: *Journ. Polymer Sci.*, **26**, 323 (1957); **36**, 455 (1959).
[18] H. F. WHITE e C. M. LOVELL: *Journ. Polymer Sci.*, **41**, 369 (1959).
[19] I. NAKAGAWA e S. I. MIZUSHIMA: *Journ. Chem. Phys.*, **21**, 2195 (1953).
[20] S. SAUTER: *Zeits. Phys. Chem.*, B **21**, 161 (1933).

In base a queste considerazioni noi prenderemo in esame diverse strutture molecolari possibili con differenti modelli di potenziale.

a) La catena è uniforme nella sua struttura angolare, le posizioni di equilibrio essendo individuate da un angolo φ_0 attorno al quale avvengono le rotazioni per effetto del moto browniano. Una forma matematicamente conveniente della barriera di potenziale è rappresentata in questo caso dal primo termine di uno sviluppo in serie di Fourier

$$U(\varphi) = \frac{U_0}{2}[1 - \cos(\varphi - \varphi_0)] \tag{11}$$

dove U_0 è l'altezza della barriera di potenziale. Il valore medio delle funzioni trigonometriche che caratterizzano la struttura angolare della catena è espresso mediante le funzioni di Bessel con argomento immaginario

$$\begin{cases} \eta = \langle \cos\varphi \rangle = \dfrac{\int_{-\pi}^{+\pi} \exp[-U(\varphi)/RT] \cos\varphi \, d\varphi}{\int_{-\pi}^{+\pi} \exp[-U(\varphi)/RT] \, d\varphi} = \cos\varphi_0 \dfrac{I_1(U_0/2RT)}{I_0(U_0/2RT)}, \\ \varepsilon = \langle \sin\varphi \rangle = \dfrac{\int_{-\pi}^{+\pi} \exp[-U(\varphi)/RT] \sin\varphi \, d\varphi}{\int_{-\pi}^{+\pi} \exp[-U(\varphi)/RT] \, d\varphi} = \sin\varphi_0 \dfrac{I_1(U_0/2RT)}{I_0(U_0/2RT)}. \end{cases} \tag{12}$$

I_1 e I_0 sono le funzioni di Bessel modificate del primo tipo rispettivamente di ordine uno e zero ([21]).

b) La catena presenta una periodicità 3, dovuta al susseguirsi ordinato delle interazioni C—C (sequenza CCOC), delle interazioni C′—C′ (sequenza COCC) e delle interazioni O—O (sequenza OCCO). Assumendo eguali le due interazioni C—C e C′—C′, e trascurando l'effetto perturbante degli estremi (interazioni H—C, anzichè C—C), possiamo porre

$$\begin{cases} U(\varphi_{3s+1}) = U(\varphi_{3s+2}) & \text{Interazioni C—C} \\ U(\varphi_{3s+3}) & \text{Interazioni O—O,} \end{cases} \tag{13}$$

dove $s = 0, 1, 2, \ldots, x$ per $(3s+1)$ e $s = 0, 1, 2, \ldots, (x-1)$ per $(3s+2)$ e $(3s+3)$.

([21]) G. N. Watson: *A Treatise on the Theory of Bessel Functions* (Cambridge, 1952).

Il valore medio quadratico del momento elettrico della molecola è allora descritto dai quattro parametri angolari

$$(14)\qquad \begin{cases} \eta_1 = \langle \cos \varphi_{3s+1} \rangle = \langle \cos \varphi_{3s+2} \rangle = \cos \varphi_1 \dfrac{I_1(U_1/2RT)}{I_0(U_1/2RT)} \\ \varepsilon_1 = \langle \sin \varphi_{3s+1} \rangle = \langle \sin \varphi_{3s+2} \rangle = \sin \varphi_1 \dfrac{I_1(U_1/2RT)}{I_0(U_1/2RT)} \\ \eta_2 = \langle \cos \varphi_{3s+3} \rangle = \cos \varphi_2 \dfrac{I_1(U_2/2RT)}{I_0(U_2/2RT)} \\ \varepsilon_2 = \langle \sin \varphi_{3s+3} \rangle = \sin \varphi_2 \dfrac{I_1(U_2/2RT)}{I_0(U_2/2RT)} \end{cases}$$

dove φ_1 è l'angolo di asimmetria per la rotazione dei momenti elementari $\boldsymbol{\mu}_{3s+2}$ e $\boldsymbol{\mu}_{3s+3}$, φ_2 l'angolo di asimmetria per la rotazione del momento elementare $\boldsymbol{\mu}_{3s+4}$, U_1 e U_2 le altezze delle rispettive barriere di potenziale del tipo (11) che si oppongono alla rotazione libera.

c) La catena è uniforme nella sua struttura angolare, tranne gli estremi la cui configurazione di equilibrio è differente. L'importanza dell'effetto degli estremi diminuisce col crescere del peso molecolare, e per un valore abbastanza grande di quest'ultimo si ricade nel caso *a*). Questo è un caso particolare di *b*).

Abbiamo allora

$$(15)\qquad \begin{cases} U(\varphi_1) = U(\varphi_{3x+1}), \\ U(\varphi_s) = U(\varphi) \qquad\qquad s = 2, 3, \ldots, 3x\,. \end{cases}$$

Anche in questo caso la media quadratica del momento elettrico è convenientemente descritta dai quattro parametri $\eta_1, \eta_2, \varepsilon_1, \varepsilon_2$ il cui significato è ormai ovvio.

Osserviamo a questo punto che, quale che sia la vera forma della barriera di potenziale effettiva, la barriera a due parametri del tipo (11) costituisce sempre un'ottima approssimazione per calcoli effettivi e può essere ottenuta come approssimazione in uno sviluppo in serie del potenziale reale.

4. – Il valor medio quadratico del momento elettrico.

Nell'ipotesi della indipendenza statistica delle rotazioni, mediamo ora la (9) su tutte le configurazioni microscopiche possibili assumendo una barriera di potenziale di tipo (11). La media è effettuata mediando i singoli elementi di matrice in accordo con i tre diversi modelli molecolari illustrati precedentemente.

4·1. *Catena uniforme.* – Possiamo porre in questo caso

$$\langle A_s \rangle = A \qquad s = 1, 2, \ldots, (3x+1) \tag{16}$$

dove si è indicata con A la matrice media

$$A = \begin{bmatrix} \alpha & -\beta\eta & \beta\varepsilon \\ -\beta & -\alpha\eta & \alpha\varepsilon \\ 0 & -\varepsilon & -\eta \end{bmatrix} \tag{17}$$

e η, ε sono dati dalle relazioni (12).

Sviluppando e sommando si ottiene

$$M = 2\mu_A^2[I - AF^{(1)}(A^3)] - 4\mu_A\mu_E(A - A^3)F^{(2)}(A^3) + \\ + 2\mu_E^2[x(I - A^2) - (I - A^2)(A - A^3)F^{(3)}(A^3)] \tag{18}$$

avendo indicato con M la matrice media il cui elemento 11 dà il momento elettrico quadratico medio

$$\overline{\mu^2} = 2\mu_A^2\,[1 - \sum_{p=1}^{3} A_{1p}F^{(1)}_{p1}(A^3)] - 4\mu_A\mu_E \sum_{p=1}^{3} (A - A^3)_{1p}\,F^{(2)}_{p1}(A^3) + \\ + 2\mu_E^2[x(I - A^2)_{11} - \sum_{p,q=1}^{3} (I - A^2)_{1p}(A - A^3)_{pq}F^{(3)}_{q1}(A^3)]\,, \tag{19}$$

e con A_{pq}, $F^{(i)}_{pq}$ gli elementi pq delle rispettive matrici. Nelle (18) e (19) si è posto

$$F^{(1)}(A^3) = A^{3x} \tag{20}$$

$$F^{(2)}(A^3) = \frac{I - A^{3x}}{I - A^3} \tag{21}$$

$$F^{(3)}(A^3) = \frac{x(I - A^3) + A^{3x} - I}{(I - A^3)^2}\,. \tag{22}$$

Se $x \gg 1$ e la catena è flessibile, allora

$$F^{(3)}(A^3) = x(I - A^3)^{-1} \tag{23}$$

e la (19) diventa

$$\overline{\mu^2} = 2\mu_E^2 x \sum_{p,q=1}^{3} (I + A)_{1q}(I + A + A^2)^{-1}_{pq}(I - A)_{q1} . \tag{24}$$

Questa espressione può essere valutata esplicitamente (vedere l'Appendice Matematica per i dettagli del calcolo) e si ottiene

$$\overline{\mu^2} = 2\mu_E^2 x(1 - \alpha^2) \frac{1 - \eta - (\eta^2 + \varepsilon^2)(\eta^2 + \varepsilon^2 - \eta)}{(1 - \eta + \eta^2)A + \varepsilon^2 B} \tag{25}$$

con

$$\begin{cases} A = (1 + \alpha + \alpha^2)(1 + \eta^2) - (2\alpha^2 + 2\alpha - 1)\eta \\ B = (1 + \alpha + \alpha^2)(2\eta^2 + \varepsilon^2) - (1 + \alpha)(2 + 3\alpha\eta) + \alpha^2 . \end{cases} \tag{26}$$

Se l'impedimento alla rotazione è simmetrico, $\varepsilon = 0$ e la (25) diventa

$$\overline{\mu^2} = 2\mu_E^2 x \frac{(1 - \alpha^2)(1 - \eta^2)}{(1 + \alpha + \alpha^2)(1 + \eta^2) - (2\alpha^2 + 2\alpha - 1)\eta} . \tag{27}$$

Se la rotazione è libera, si deve porre $U_0 = 0$ nell'equazione (11) e la (19) diventa semplicemente

$$\overline{\mu^2} = 2\mu_A^2(1 - \alpha^{3x+1}) - 4\mu_A \mu_E(\alpha - \alpha^3) \frac{1 - \alpha^{3x}}{1 - \alpha^3} + \tag{28}$$

$$+ 2\mu_A^2 \left[x(1 - \alpha^2) - (1 - \alpha^2)(\alpha - \alpha^3) \frac{x(1 - \alpha^3) + \alpha^{3x} - 1}{(1 - \alpha^3)^2} \right],$$

che coincide col risultato di KUBO [3].

4·2. *Catena periodica con periodo* 3. – In questo caso la catena ha una periodicità del tipo

$$(+)(+)(-)(+)(+)(-)(+)(+)(-) \ldots$$

e si può porre

$$\begin{cases} \langle A_{3s+1} \rangle = \langle A_{3s+2} \rangle = R_1 & (+) \\ \langle A_{3s+3} \rangle = R_2 & (-) \end{cases} \tag{29}$$

con $s = 0, 1, 2, \ldots, x$ per $(3s + 1)$; $s = 0, 1, 2, \ldots, (x - 1)$ per $(3s + 2)$ e $(3s + 3)$.

R_1 e R_2 sono date rispettivamente dalle relazioni

$$R_1 = \begin{bmatrix} \alpha & -\beta\eta_1 & \beta\varepsilon_1 \\ -\beta & -\alpha\eta_1 & \alpha\varepsilon_1 \\ 0 & -\varepsilon_1 & -\eta_1 \end{bmatrix} \tag{30}$$

$$R_2 = \begin{bmatrix} \alpha & -\beta\eta_2 & \beta\varepsilon_2 \\ -\beta & -\alpha\eta_2 & \alpha\varepsilon_2 \\ 0 & -\varepsilon_2 & -\eta_2 \end{bmatrix} \tag{31}$$

con $\eta_1, \eta_2, \varepsilon_1, \varepsilon_2$ dati dalle (14).

Conviene introdurre le due matrici S e T definite da

$$S = R_1 R_2 R_1 \tag{32}$$

$$T = R_2 R_1 R_1 . \tag{33}$$

Le matrici S e T soddisfano alle seguenti relazioni

$$\begin{cases} R_1 T = S R_1 \\ T R_2 R_1 = R_2 R_1 S \\ R_1 F(T) = F(S) R_1 . \end{cases} \tag{34}$$

Rimandiamo all'Appendice Matematica per il dettaglio dei calcoli.

Tenendo conto delle (34), introducendo le matrici R_1, R_2, S e T nella (9), sviluppando e sommando, si ottiene

$$M = 2\mu_A^2 (I - S^x R_1) - 2\mu_A \mu_E \left[(R_1 - S) \frac{I - S^x}{I - S} + (R_1 - T) \frac{I - T^x}{I - T} \right] + \\ + 2\mu_E^2 \{ x(I - R_2 R_1) - (I - R_2 R_1)[R_1 F(T) - S F(S)] \} , \tag{35}$$

$$\overline{\mu^2} = M_{11} . \tag{36}$$

Abbiamo posto per brevità

$$\begin{cases} F(S) = \dfrac{x(I - S) + S^x - I}{(I - S)^2} , \\ F(T) = \dfrac{x(I - T) + T^x - I}{(I - T)^2} . \end{cases} \tag{37}$$

Se $x \gg 1$ e la catena è flessibile, si ha la formula approssimata

$$\frac{\overline{\mu^2}}{2\mu_E^2 x} = \sum_{p,q=1}^{3} (I - R_2 R_1)_{1p} (I - S)^{-1}_{pq} (I - R_1)_{q1}, \tag{38}$$

che può essere valutata esplicitamente nel caso di rotazioni impedite simmetriche. In tal caso $\varepsilon_1 = \varepsilon_2 = 0$, e si ottiene

$$\frac{\overline{\mu^2}}{2\mu_E^2 x} = \frac{(1-\alpha^2)(1-\eta_1^2)(1-\eta_2)}{(1-\eta_2)[(1+\alpha+\alpha^2)(1+\eta_1^2) - (2\alpha^2+2\alpha-1)\eta_1] - (1+\eta_1)(\eta_1-\eta_2)}. \tag{39}$$

Per $\eta_1 = \eta_2 = \eta$ questa formula ridà la (27).

Il caso generale ($\varepsilon_1 \neq 0$, $\varepsilon_2 \neq 0$) può essere convenientemente studiato introducendo i valori numerici direttamente nella equazione (38).

Se nella (35) si pone

$$\begin{cases} R_1 = R_2 = A \\ S = T = A^3 \end{cases} \tag{40}$$

si riottengono le formule (18), (19), (24), (25) del caso della catena uniforme.

4'3. *Catena uniforme tranne gli estremi.* – La struttura è del tipo

$$(+)(-)(-)(-)(-) \ldots (-)(+)$$

e poniamo

$$\begin{cases} \langle A_1 \rangle = \langle A_{3x+1} \rangle = R_1 \\ \langle A_s \rangle = R_2 \qquad\qquad s = 2, 3, \ldots, 3x \end{cases} \tag{41}$$

dove R_1 e R_2 sono date da espressioni analoghe alle (30) e (31). A calcoli fatti si ottiene

$$M = 2\mu_A^2 (I - R_1 R_2^{3x-1} R_1) - 4\mu_A \mu_E R_1 (I - R_2^2) \frac{I - R_2^{3x}}{I - R_2^3} + \tag{42}$$

$$+ 2\mu_E^2 [x(I - R_2^2) - (I - R_2^2)(R_2 - R_2^3) F(R_2^3)],$$

$$\overline{\mu^2} = M_{11} \tag{43}$$

che si riduce alla (18) per $R_1 = R_2 = A$, e alla (24) per grandi valori di x.

Si è posto nella (42)

$$F(R_2^3) = \frac{x(I - R_2^3) + R_2^{3x} - I}{(I - R_2^3)^2}. \tag{44}$$

5. – La valutazione esatta del valore medio quadratico del momento elettrico.

Si può subito rilevare la complessità delle formule esatte (18), (35) e (42) anche se il numero di parametri necessari per descrivere la configurazione media della catena è molto limitato. La complessità è dovuta soprattutto all'effetto dei gruppi alcoolici terminali, che è sensibile per piccoli valori di x. È però possibile anche in questo caso il calcolo numericamente esatto del momento elettrico quadratico medio, ricorrendo a metodi di interpolazione ([10,12]).

In quanto segue, ci riferiremo come ad un esempio caratteristico all'equazione (18).

Posto $A^3 = X$, la risoluzione dell'equazione agli autovalori

$$|X - \lambda I| = \lambda^3 - \sigma_1\lambda^2 + \sigma_2\lambda - \sigma_3 = 0 \tag{45}$$

permette di trovare le tre radici $\lambda_1, \lambda_2, \lambda_3$ (autovalori), noti gli invarianti $\sigma_1, \sigma_2, \sigma_3$ della matrice X.

Se $\varepsilon \neq 0$, le radici dell'equazione agli autovalori sono una reale e due complesse coniugate, quando l'angolo di valenza è 109° 28′.

Per $\varepsilon = 0$ si ha

$$\begin{cases} \sigma_1 = \alpha(1-\eta)[3\eta + \alpha^2(1-\eta)^2] - \eta^3 \\ \sigma_2 = -\eta^3\{1 + \alpha(1-\eta)[3\eta + \alpha^2(1-\eta)^2]\} \\ \sigma_3 = \eta^6 \end{cases} \tag{46}$$

e una radice è data da $\lambda_3 = -\eta^3$, mentre λ_1 e λ_2 si trovano risolvendo l'equazione di secondo grado in λ

$$\lambda^2 - \alpha(1-\eta)[3\eta + \alpha^2(1-\eta)^2]\lambda - \eta^3 = 0\,. \tag{47}$$

In questo caso, per $\varphi_0 = 0$ si hanno tre radici reali, per $\varphi_0 = \pi$ una radice è reale ($\lambda_3 = -\eta^3$) e le altre due complesse coniugate.

Note $\lambda_1, \lambda_2, \lambda_3$ possiamo rappresentare ogni funzione $F^{(i)}(X)$ $(i = 1, 2, 3)$ della matrice X nella forma di Lagrange

$$F^{(i)}(X) = \sum_{(l,m,n)} \frac{(X - \lambda_l I)(X - \lambda_m I)}{(\lambda_n - \lambda_l)(\lambda_n - \lambda_m)} F^{(i)}(\lambda_n) = a_i X^2 + b_i X + c_i I\,, \tag{48}$$

dove i coefficienti a_i, b_i, c_i sono dati dalle relazioni

$$\begin{cases} a_i = \sum_{(l,m,n)} \dfrac{1}{(\lambda_n - \lambda_l)(\lambda_n - \lambda_m)} F^{(i)}(\lambda_n)\,, \\ b_i = \sum_{(l,m,n)} \dfrac{-(\lambda_l + \lambda_m)}{(\lambda_n - \lambda_l)(\lambda_n - \lambda_m)} F^{(i)}(\lambda_n)\,, \\ c_i = \sum_{(l,m,n)} \dfrac{\lambda_l \lambda_m}{(\lambda_n - \lambda_l)(\lambda_n - \lambda_m)} F^{(i)}(\lambda_n)\,, \end{cases} \tag{49}$$

dove (l, m, n) indica una permutazione ciclica degli indici. Noti gli elementi (pq) delle matrici $F^{(i)}(X)$, la (19) permette il calcolo numerico di $\overline{\mu^2}$.

Calcoli numerici per i diversi modelli molecolari, per differenti valori delle barriere di potenziale, delle asimmetrie alla rotazione e del numero x di periodi della catena, sono in corso presso questo Laboratorio con un calcolatore elettronico IBM-650, come già accennato nell'introduzione. Rimandiamo pertanto la discussione quantitativa dei risultati alla parte II di questa nota.

Appendice matematica

Sviluppiamo in maggior dettaglio i calcoli relativi alla catena periodica con periodo tre (caso b) di Sezione 4).

L'equazione (19) può scriversi

$$
\begin{aligned}
\text{(A-1)}\quad \overline{\mu^2} = {} & (\mu_A, 0, 0)(I - A_{3x+1}A_{3x} \dots A_1)(I - A_1^* \dots A_{3x}^* A_{3x+1}^*)(\mu_A, 0, 0)^* + \\
& + (\mu_A, 0, 0)(I - A_{3x+1}A_{3x} \dots A_1)[(A_1^* A_2^* A_3^* + A_1^* \dots A_6^* + \dots + A_1^* \dots A_{3x}^*) - \\
& - (A_1^* + A_1^* \dots A_4^* + \dots + A_1^* \dots A_{3x-2}^*)](\mu_E, 0, 0)^* + \\
& + (\mu_E, 0, 0)[(A_3 A_2 A_1 + A_6 \dots A_1 + \dots + A_{3x} \dots A_1) - \\
& - (A_1 + A_4 \dots A_1 + \dots + A_{3x-2} \dots A_1)](I - A_1^* \dots A_{3x+1}^*)(\mu_A, 0, 0)^* + \\
& + (\mu_E, 0, 0)[(A_3 A_2 A_1 + A_6 \dots A_1 + \dots + A_{3x} \dots A_1) - \\
& - (A_1 + A_4 \dots A_1 + \dots + A_{3x-2} \dots A_1)][(A_1^* A_2^* A_3^* + A_1^* \dots A_6^* + \dots + A_1^* \dots A_{3x}^*) - \\
& - (A_1^* + A_1^* \dots A_4^* + \dots + A_1^* \dots A_{3x-2}^*)](\mu_E, 0, 0)^* = B + C + C^* + D\,.
\end{aligned}
$$

Moltiplichiamo ed eseguiamo le medie in conformità delle (29), introducendo le matrici S e T e tenendo presenti le relazioni (7) e (8). Abbiamo per i singoli contributi

$$
\text{(A-2)}\qquad \frac{B}{\mu_A^2} = I - S^x R_1 - R_1^* S^{*x} + I = 2(I - S^x R_1)
$$

$$
\begin{aligned}
\text{(A-3)}\qquad \frac{C + C^*}{\mu_A \mu_E} = {} & T^*(I + T^* + \dots + T^{*x-1}) + S(I + S + \dots + S^{x-1}) - \\
& - R_1(I + T + \dots + T^{x-1}) - R_1^*(I + S^* + \dots + S^{*x-1}) + \\
& + T(I + T + \dots + T^{x-1}) + S^*(I + S^* + \dots + S^{*x-1}) - \\
& - R_1^*(I + S^* + \dots + S^{*x-1}) - R_1(I + T + \dots + T^{x-1}) = \\
& = -2\left[(R_1 - S)\frac{I - S^x}{I - S} + (R_1 - T)\frac{I - T^x}{I - T}\right].
\end{aligned}
$$

$$\text{(A-4)}\quad \frac{D}{\mu_E^2} = xI + 2T[(x-1)I + (x-2)T + \ldots + T^{x-2}] +$$
$$+ xI + 2S[(x-1)I + (x-2)S + \ldots + S^{x-2}] -$$
$$- \{xI + S^*[(x-1)I + (x-2)S^* + \ldots + S^{*x-2}]\} R_1^* R_2^* -$$
$$- R_1[(x-1)I + (x-2)T + \ldots + T^{x-2}] -$$
$$- R_2 R_1 \{xI + S[(x-1)I + (x-2)S + \ldots + S^{x-2}]\} -$$
$$- [(x-1)I + (x-2)T^* + \ldots + T^{*x-2}]R_1^* =$$
$$= 2\{x(I - R_2 R_1) - (I - R_2 R_1)[R_1 F(T) - S F(S)]\}\,.$$

La somma di (A-2), (A-3), (A-4) fornisce l'equazione (35) del testo. Analogamente si procede per la (42).

Se ora poniamo $R_1 = R_2 = A$, $S = T = A^3$, otteniamo nel caso limite di $x \gg 1$ e $|\lambda|^x < 1$ l'equazione (24)

$$\frac{\overline{\mu^2}}{2\mu_E^2 x} = \sum_{p,q=1}^{3} (I + A)_{1p}(I + A + A^2)^{-1}_{pq}(I - A)_{q1}\,.$$

L'espressione esplicita del prodotto è

$$\text{(A-5)}\quad [1+\alpha, -\beta\eta, \beta\varepsilon]\begin{bmatrix} 1+\alpha+\alpha^2+\beta^2\eta & -\beta\eta+\beta(\sigma-\alpha\eta) & \beta\varepsilon+\beta(\alpha\varepsilon-\tau) \\ -\beta-\alpha\beta(1-\eta) & 1-\alpha\eta+\beta^2\eta+\alpha\sigma & \alpha\varepsilon-(\beta^2\varepsilon+\alpha\tau) \\ \beta\varepsilon & \tau-\varepsilon & 1-\eta+\varrho \end{bmatrix}^{-1}\begin{pmatrix} 1-\alpha \\ \beta \\ 0 \end{pmatrix}$$

con

$$\text{(A-6)}\qquad \sigma = \alpha\eta^2 - \varepsilon^2\,, \qquad \tau = (1+\alpha)\eta\varepsilon\,, \qquad \varrho = \eta^2 - \alpha\varepsilon^2\,.$$

È possibile mediante opportuni accorgimenti [9,12], trasformare (A-5) nel seguente prodotto

$$\beta^2(1,0,0)\left\{\begin{bmatrix} -1 & 0 & 0 \\ \beta & -(1-\alpha) & 0 \\ 0 & 0 & 1-\alpha \end{bmatrix}\cdot\right.$$
$$\cdot\begin{bmatrix} 1+\alpha+\alpha^2+\beta^2\eta & -\beta\eta+\beta(\sigma-\alpha\eta) & \beta\varepsilon+\beta(\alpha\varepsilon-\tau) \\ -\beta-\alpha\beta(1-\eta) & 1-\alpha\eta+\beta^2\eta+\alpha\sigma & \alpha\varepsilon-(\beta^2\varepsilon+\alpha\tau) \\ \beta\varepsilon & \tau-\varepsilon & 1-\eta+\varrho \end{bmatrix}\cdot$$
$$\left.\cdot\begin{bmatrix} -1 & -\beta\eta & -\beta\varepsilon \\ 0 & -(1+\alpha) & 0 \\ 0 & 0 & 1+\alpha \end{bmatrix}\right\}^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Eseguendo il prodotto tra parentesi a graffa con le regole di moltiplicazione delle matrici, si ha

$$\frac{\overline{\mu^2}}{2\mu_E^2 x} = \beta^2 \begin{bmatrix} 1+\alpha+\alpha^2+\beta^2\eta & \alpha\beta(\eta^2-\varepsilon^2-\eta)+\beta^2(\eta^2-\varepsilon^2) & 2\beta\eta\varepsilon-\alpha\beta(1-2\eta)\varepsilon \\ -\beta(2+\alpha+\eta-\alpha\eta) & -\beta^2(\eta^2-\varepsilon^2-1) & -2\beta^2\eta\varepsilon \\ -\beta(1-\alpha)\varepsilon & \beta^2(1-2\eta)\varepsilon & \beta^2(\eta^2-\varepsilon^2-\eta+1) \end{bmatrix}^{-1}_{11}$$

da cui è facile risalire alla equazione (25) del testo, una volta calcolato il determinante.

* * *

Desidero ringraziare vivamente il Prof. Corrado Rossi, il Prof. Antonio Borsellino e i Colleghi dell'Istituto di Chimica Industriale per le utili discussioni avute durante la preparazione di questo lavoro.

SUMMARY

The mean square electric dipole moment for polyoxyethyleneglycol has been calculated by matrices of rotation method for three different molecular structures, supposing the internal flexibility of the macromolecule in solution and statistical independence of rotations. The three models are: 1) the chain with statistically uniform internal rotations; 2) the periodic chain with period three; 3) the uniform chain with different rotations at two ends. The internal rotations, which can be also asymmetric in respect to the « trans » configuration, are hindered by a periodic potential barrier such as $U(\varphi)=(U_0/2)[1-\cos(\varphi-\varphi_0)]$. Explicit formulae are given for the three molecular models in the case of a sufficient large number x of periods of the chain. For any value of x it is possible to calculate numerically the mean square electric dipole moment $\overline{\mu^2}$ by means of the Lagrange representation formula.

On the Coulomb Scattering by a Spherical Charge Distribution.

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(ricevuto il 25 Luglio 1960)

Summary. — An analytic expression for the total scattering amplitude of charged particles by a static charge distribution with spherical symmetry, has been derived. The corresponding potential is supposed to be an arbitrary superposition of Laguerre functions. The evaluation has been performed using the Coulomb point-charge waves as an approximation of first order, and iterating one time for taking into account the charge's extension.

1. – Introduction.

Recently a large number of scattering experiments with beams of electrons of high energy (several MeV) ([1-5]) have been performed in order to get information on the nuclear electric charge distribution.

The theoretical interpretation of the diffraction curves is made treating the nucleus as a static charge distribution, in most cases of spherical symmetry. One admits from the very beginning a particular shape of this distribution and one carries out cumbersome numerical calculations at each given energy of the many phases which enter in a partial wave expansion of the total scattering amplitude.

([1]) R. Hofstadter: *Rev. Mod. Phys.*, **28**, 214 (1956).
([2]) B. Hahn, R. Hofstadter and D. G. Ravenhall: *Phys. Rev.*, **105**, 1353 (1957).
([3]) G. R. Burleson and R. Hofstadter: *Phys. Rev.*, **112**, 1282 (1958).
([4]) R. Hofstadter, F. Bumiller and M. R. Yearian: *Rev. Mod. Phys.*, **30**, 482 (1958).
([5]) B. Hahn, D. G. Ravenhall and R. Hofstadter: *Phys. Rev.*, **101**, 1131 (1956.)

We wish to present here some evaluation of the same scattering amplitude without specifying *a priori* the shape of the potential but letting it be an arbitrary superposition of Laguerre functions. Since they form a complete basis, any potential can be thought of as a particular superposition of them. There is therefore no restriction on the form assumed for the static potential.

Our treatment of course cannot be based on a partial wave expansion and has been centered about the possibility of obtaining closed expressions for the total scattering amplitude, by means of a kind of Born approximation in which the unperturbed wave function is that corresponding to the Coulomb potential of a point charge.

Since in the near future more experiments will be performed on the same subject, making use also of new particles, *e.g.* μ-mesons ($(1 \div 5)$GeV) as projectiles we retain that our type of calculations will be of good help for the corresponding interpretation of the experimental data.

In Section 2 we shall give the form of the potential and determine the analytic expression of the scattering amplitude, while in Section 3 we shall consider the proton as a particular case and treat also the neutron with the Born approximation.

2. – Evaluation of the scattering amplitude.

We suppose the nucleus to consist of a charge distribution whose potential is finite everywhere and corresponds asymptotically to the Coulomb potential and is completely arbitrary for what concerns its behaviour at finite distances. The potential may thus be assumed to be of the form

$$A\left[\frac{1-\exp\left[-\beta r\right]}{\beta r}+f(r)\right], \tag{1}$$

where the function $f(r)$ has the only purpose of modifying the charge distribution and must consequently satisfy the condition of corresponding to a vanishing total charge

$$\int_0^\infty A^2 f(r)\,\mathrm{d}^3 r = 4\pi\left[r^2\frac{\mathrm{d}f}{\mathrm{d}r}\right]_0^\infty = 0\,. \tag{2}$$

With our assumptions such a function is expressible as a series in terms of a complete set of orthogonal functions in the interval $(0\ \infty)$ as for instance the Laguerre functions

$$f(r) = \sum c_m \exp\left[-\frac{\varepsilon}{2}r\right] L_m(\varepsilon r)\,. \tag{3}$$

In this case (1) represents a potential of the required type, corresponding to a total charge Ze after a suitable choice of the constant A. We shall write then the potential in the form

$$V(r) = A\left[\frac{1-\exp[-\beta r]}{\beta r} + \sum c_m \exp\left[-\frac{\varepsilon}{2}r\right] L_m(\varepsilon r)\right]. \tag{4}$$

Since we consider high energies it is allowed to assume that the impinging particles satisfy the spinless Klein-Gordon equation

$$(p^2c^2 + m^2c^4)\psi = (E - V)\psi\,,$$

which neglecting $(V/E)^2$ with respect to $2V/E$ may be written as follows:

$$\Delta^2\psi + (k^2 - U(r))\psi = 0\,, \tag{5}$$

where

$$k^2 = \frac{E^2 - m^2c^4}{\hbar^2c^2} \simeq \frac{E^2}{\hbar^2c^2}\,, \tag{6}$$

$$U(r) = \frac{2Z\beta Eeq}{\hbar^2c^2}\left[\frac{1-\exp[-\beta r]}{\beta r} + \sum c_m \exp\left[-\frac{\varepsilon}{2}r\right] L_m(\varepsilon r)\right] = U^C(r) + \Omega(r)\,, \tag{7}$$

and q is the charge of the impinging particle ($\pm e$) while $U^C(r)$ is the Coulomb potential. Equation (5) may then be written

$$\Delta^2\psi + (k^2 - U^C(r))\psi = \Omega(r)\,\psi\,. \tag{8}$$

Putting $\psi = L + \varphi$, where L satisfies the equation

$$\Delta^2 L + (k^2 - U^C(r))\psi = 0\,, \tag{9}$$

one gets [6]

$$\varphi(\boldsymbol{r}) = \int \mathrm{d}^3r'\, K(\boldsymbol{r}, \boldsymbol{r}')\,\Omega(r')\,\psi(\boldsymbol{r}')\,, \tag{10}$$

with

$$K(\boldsymbol{r}, \boldsymbol{r}') = \begin{cases} -\dfrac{k}{4\pi}\displaystyle\sum_{n=0}^{\infty}(2n+1)\,A_n(r)\,H_n(r')\,P_n(\cos\Theta)\,; & r' > r, \\ -\dfrac{k}{4\pi}\displaystyle\sum_{n=0}^{\infty}(2n+1)\,A_n(r')\,H_n(r)\,P_n(\cos\Theta)\,; & r' < r, \end{cases} \tag{11}$$

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[6] N. F. Mott and H. S. W. Massey: *The Theory of Atomic Collisions*, 2nd ed., p. 112

and

$$(12)\qquad \begin{cases} \cos\Theta = \cos(\boldsymbol{r}, \boldsymbol{r}') = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi' - \varphi)\,, \\ \cos\theta = \cos(\boldsymbol{r}, \boldsymbol{k})\,; \qquad \cos\theta' = \cos(\boldsymbol{r}', \boldsymbol{k})\,. \end{cases}$$

Λ_n and H_n satisfy the radial part of equation (9) and are characterized by the following asymptotic behaviour

$$(13)\qquad \begin{cases} \Lambda_n \sim \dfrac{1}{kr}\sin\left(kr - \dfrac{1}{2}n\pi + \delta_n - \lambda\ln 2kr\right) \\ H_n \sim \dfrac{1}{kr}\exp\left[i\left(kr - \dfrac{1}{2}n\pi + \delta_n - \lambda\ln 2kr\right)\right] \end{cases} \qquad \lambda = \frac{Zeq}{\hbar v}\,.$$

Because of (13) the asymptotic behaviour of the kernel in equation (10) is

$$(14)\qquad K(\boldsymbol{r}, \boldsymbol{r}') \underset{r\to\infty}{\sim} -\frac{\exp[ikr]}{4\pi r}\exp[-i\lambda\ln 2kr]\mathscr{F}(r, \pi - \Theta)\,.$$

$\mathscr{F}(r, \theta)$ is the exact solution of equation (9) and is represented by the following expression (7):

$$(15)\qquad \mathscr{F}(r, \theta) = \sum_{n=0}^{\infty}(2n+1)(i)^n\exp[i\delta_n]\Lambda_n(r)P_n(\cos\theta) = L(r, \theta) =$$
$$= \Gamma(1 + i\lambda)\exp[ikr\cos\theta]\exp\left[-\frac{\pi}{2}\lambda\right]F[-i\lambda, 1; ikr(1 - \cos\theta)]\,,$$

where F is the regular confluence hypergeometric function.

Then

$$(16)\qquad \psi_{\sim} = L_{\sim} - \frac{\exp[-i\lambda\ln 2kr + ikr]}{4\pi r}\int \mathrm{d}^3 r'\, L(r', \pi - \Theta)\,\Omega(r')\,\psi(\boldsymbol{r}')\,,$$

where

$$(17)\qquad L_{\sim} = \exp\left[ikr\cos\theta + i\lambda\ln\left[2kr\sin^2\frac{\theta}{2}\right]\right] +$$
$$+ \frac{\Gamma(1 + i\lambda)}{\Gamma(-i\lambda)}\cdot\frac{\exp[ikr - i\lambda\ln(2kr\sin^2\theta/2)]}{2ikr\sin^2\theta/2}\,.$$

In a first approximation we replace $\psi(\boldsymbol{r}')$ on the right hand side of (16) by

(7) H. A. Bethe and L. C. Maximon: *Phys. Rev.*, **93**, 768 (1954).

$L(r', \theta')$ so that the integral in (16) becomes

$$(18) \qquad I = \exp\left[-\pi\lambda\right]\Gamma^2(1+i\lambda)\int_0^{2\pi}\mathrm{d}\varphi'\int_0^{\pi}\sin\theta'\,\mathrm{d}\theta'\int_0^{\infty}r'^2\mathrm{d}r'\exp\left[ikr'(\cos\Theta-\cos\theta')\right]\cdot$$

$$\cdot F[-i\lambda, 1; ikr'(1+\cos\Theta)]\cdot\Omega(r')\cdot F[-i\lambda, 1; ikr'(1-\cos\theta')]\,.$$

Using parabolic co-ordinates

$$(19) \qquad \begin{cases} \xi = r'(1+\cos\theta')\,, \\ \eta = r'(1-\cos\theta')\,, \\ r' = \dfrac{\xi+\eta}{2}\,, \end{cases}$$

$$(20) \qquad \mathrm{d}^3r' = \tfrac{1}{4}(\xi+\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\varphi'$$

and putting

$$(21) \qquad u = ikr'(1+\cos\Theta) = ik\left[\frac{\xi+\eta}{2}+\cos\theta\,\frac{\xi-\eta}{2}+\sin\theta\cos(\varphi'-\varphi)\sqrt{\xi\eta}\right],$$

$$(22) \qquad \theta = \pi+\alpha\,,$$

we obtain

$$(23) \qquad u = ik\left[\xi\sin^2\frac{\alpha}{2}+\eta\cos^2\frac{\alpha}{2}-2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\sqrt{\xi\eta}\cos(\varphi'-\varphi)\right],$$

so that

$$(24) \qquad I = \frac{\exp\left[-\pi\lambda\right]\Gamma(1+i\lambda)}{4}\int_0^{2\pi}\mathrm{d}\varphi'\int_0^{\infty}\mathrm{d}\xi\int_0^{\infty}\mathrm{d}\eta\,(\xi+\eta)\exp\left[ik\xi\right]\cdot$$

$$\cdot F(-i\lambda, 1; ik\eta)\,\Omega\left(\frac{\xi+\eta}{2}\right)\int_0^{\infty}x^{i\lambda}\exp\left[-x\right]J_0\left(\sqrt{4xu}\right)\mathrm{d}x\,,$$

where use has ben made of the following integral representation [8]

$$(25) \qquad \exp\left[-u\right]F(-i\lambda, 1; u) = \frac{1}{\Gamma(1+i\lambda)}\int_0^{\infty}x^{i\lambda}\exp\left[-x\right]J_0\left(\sqrt{4xu}\right)\mathrm{d}x\,.$$

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[8] A. SOMMERFELD: *Ann. d. Phys.*, **11**, 257 (1931).

By performing first the integral with repsect to φ' one obtains

$$\int_0^{2\pi} \mathrm{d}\varphi' \, J_0(\sqrt{4x u}) = 2\pi J_0(\varrho) J_0(\sigma) ,$$

with

$$\varrho = 2\sqrt{ikx\xi} \sin\frac{\alpha}{2} ,$$

$$\sigma = 2\sqrt{ikx\eta} \cos\frac{\alpha}{2} . \tag{26}$$

Making still use of (25) for $F(-i\lambda, 1; ik\eta)$ (24) becomes

$$I = \frac{\pi}{2} \exp[-\pi\lambda] \int_0^\infty \mathrm{d}x \, x^{i\lambda} \exp[-x] \int_0^\infty \mathrm{d}y \, y^{i\lambda} \exp[-y] \int_0^\infty \mathrm{d}\xi \int_0^\infty \mathrm{d}\eta (\xi+\eta) \cdot \tag{27}$$

$$\cdot \Omega\left(\frac{\xi+\eta}{2}\right) \exp[ik(\xi+\eta)] J_0(2\sqrt{ik\eta y}) J_0\left(2\sqrt{ik\eta x} \cos\frac{\alpha}{2}\right) J_0\left(2\sqrt{ikx\xi} \sin\frac{\alpha}{2}\right).$$

Owing to the particular form (6) of the potential it is only necessary to evaluate the integral I replacing $(\xi+\eta)\Omega((\xi+\eta)/2)$ with an expression of the type

$$(\xi+\eta)^n \exp[-\mu(\xi+\eta)] .$$

The integral I will then be a linear combination (with suitable coefficients) of integrals of the following type (9):

$$I_n^{(\mu)} = \frac{\pi}{2} \exp[-\pi\lambda] \sum_{h=0}^{n} \binom{n}{h} \int_0^\infty \mathrm{d}x \, x^{i\lambda} \exp[-x] \int_0^\infty \mathrm{d}y \, y^{i\lambda} \exp[-y] \cdot$$

$$\cdot \int_0^\infty \mathrm{d}\xi \, \xi^h \exp[-(\mu-ik)\xi] J_0\left(2\sqrt{ikx \sin^2\frac{\alpha}{2}} \, \xi^{\frac{1}{2}}\right) \int_0^\infty \mathrm{d}\eta \, \eta^{n-h} \exp[-(\mu-ik)\eta] \cdot$$

$$\cdot J_0(2\sqrt{iky}\,\eta^{\frac{1}{2}}) J_0\left(2\sqrt{ikx \cos^2\frac{\alpha}{2}} \, \eta^{\frac{1}{2}}\right) = (-1)^n \frac{\pi}{2} \exp[-\pi\lambda] \sum_{h=0}^{n} \binom{n}{h} \cdot$$

$$\cdot \int_0^\infty \mathrm{d}x \, x^{i\lambda} \exp[-x] \int_0^\infty \mathrm{d}y \, y^{i\lambda} \exp[-y] \cdot D_\mu^{(h)} \left\{ \frac{1}{\mu-ik} \exp\left[-ikx \sin^2\frac{\alpha}{2} \Big/ (\mu-ik)\right] \right\} \cdot$$

$$\cdot D_\mu^{(n-h)} \left\{ \frac{1}{\mu-ik} \exp\left[-ik\left(y + x\cos^2\frac{\alpha}{2}\right) \Big/ (\mu-ik)\right] \cdot J_0\left(\frac{2k\cos(\alpha/2)\sqrt{xy}}{\mu-ik}\right) \right\},$$

(9) A. Erdely Editor: *Tables of Integral Transforms*, vol. II (New York, 1954) p. 125.

where

$$D_{\mu}^{(h)} = \left(\frac{\partial}{\partial \mu}\right)^{h}.$$

And by the Leibnitz formula

$$(28) \qquad I_n^{(\mu)} = \frac{\pi}{2} \exp\left[-\pi\lambda\right](-1)^n D_{\mu}^{(n)} \left\{ \frac{1}{(\mu - ik)^2} \int_0^{\infty} \mathrm{d}x\, x^{i\lambda} \exp\left[-x\,\frac{\mu(\mu+ik)}{\mu^2+k^2}\right] \cdot \right.$$

$$\left. \cdot \int_0^{\infty} \mathrm{d}y\, y^{i\lambda} \exp\left[-y\,\frac{\mu(\mu+ik)}{\mu^2+k^2}\right] \cdot J_0\left(\frac{2k\sqrt{xy}\,\cos\,(\alpha/2)(\mu+ik)}{\mu^2+k^2}\right)\right\}.$$

Assuming as new variable $z = (\mu(\mu+ik)/(\mu^2+k^2))y$ and deforming the path of integration so that it coincides with the real z axis, and using again (25), (28) becomes

$$(29) \qquad I_n^{(\mu)} = \frac{\pi}{2} \exp\left[-\pi\lambda\right](-1)^n D_{\mu}^{(n)} \left[\frac{1}{(\mu - ik)^2} \left(\frac{\mu(\mu+ik)}{\mu^2+k^2}\right)^{-(1+i\lambda)} \right] \Gamma(1+i\lambda)\, a^{-\frac{1}{2}} \cdot$$

$$\cdot \int_0^{\infty} \mathrm{d}x\, x^{(i\lambda+\frac{1}{2})-1} M_{i\lambda+\frac{1}{2},0}(ax) \exp\left[-\nu x\right],$$

with

$$(30) \qquad \begin{cases} a = \dfrac{k^2}{\mu} \cos^2 \dfrac{\alpha}{2} \dfrac{\mu+ik}{\mu^2+k^2}, \\ \nu = \dfrac{1}{2}\, a + \mu\, \dfrac{\mu+ik}{\mu^2+k^2}. \end{cases}$$

From (29) one gets

$$I_n^{(\mu)} - \frac{\pi}{2} \exp\left[-\pi\lambda\right](-1)^n D_{\mu}^{(n)} \left[\frac{1}{(\mu - ik)^2} \left(\frac{\mu(\mu+ik)}{\mu^2+k^2}\right)^{-(1+i\lambda)} \Gamma^2(1+i\lambda) \cdot \right.$$

$$\left. \cdot (\nu + \tfrac{1}{2}a)^{-(1+i\lambda)} F\left(-i\lambda,\, i\lambda+1;\, 1;\, \frac{a}{\nu+\frac{1}{2}a}\right)\right].$$

Finally by (22) and (30)

$$(31) \qquad I_n^{(\mu)} = \frac{\pi}{2} \exp\left[-\pi\lambda\right](-1)^n \Gamma^2(1+i\lambda)\, D_{\mu}^{(n)} \left[(\mu - ik)^{2i\lambda} \left(\mu^2 + k^2 \sin^2\frac{\theta}{2}\right)^{-(1+i\lambda)} \cdot \right.$$

$$\left. \cdot F\left(-i\lambda,\, i\lambda+1;\, 1;\, \frac{k^2 \sin^2(\theta/2)}{\mu^2 + k^2 \sin^2(\theta/2)}\right)\right].$$

The term containing $\exp[-\beta r]/\beta r$ in (7) (neglecting a phase factor which multiplies every partial scattering amplitude) gives to the total scattering amplitude the contribution

$$f_1(\theta) = \frac{ZeEq}{\pi\hbar^2 c^2} I_0^{(\beta/2)} ; \tag{32}$$

the contribution of the series in (4) may be written as follows

$$f_2(\theta) = -\frac{ZeqE\beta}{2\pi\hbar^2 c^2}\left[\sum_n c_n\Big(\sum_{m=0}^{n} 2^{-m} d_m I_{m+1}^{(\varepsilon/4)}\Big)\right], \tag{33}$$

where the coefficients d_m are defined by

$$L_n(\varepsilon r) = \sum_{m=0}^{n} d_m(\varepsilon r)^m . \tag{34}$$

The Coulomb scattering amplitude is

$$f_C(\theta) = -\lambda \frac{\Gamma(1+i\lambda)}{\Gamma(1-i\lambda)} \frac{\exp[-i\lambda \ln \sin^2(\theta/2)]}{2k \sin^2(\theta/2)} . \tag{35}$$

The total scattering amplitude is thus

$$f = f_C + f_1 + f_2 . \tag{36}$$

3. – Some particular examples.

When $Z = 1$ (36) is the scattering amplitude by protons. For high energies of the impinging particles $\lambda = \pm 1/137$. Expanding the hypergeometric function as a power series in λ and keeping only the first order term one gets

$$I_n^{(\mu)} = \frac{\pi}{2} \exp[-\pi\lambda]\Gamma^2(1+i\lambda)(-1)^n D_\mu^{(n)}\left[(\mu - ik)^{2i\lambda}\left(\mu^2 + k^2 \sin^2\frac{\theta}{2}\right)^{-(1+i\lambda)} \cdot \left\{1 - i\lambda \ln\left(1 + \frac{k^2}{\mu^2}\sin^2\frac{\theta}{2}\right)\right\}\right] . \tag{37}$$

For what concerns the scattering by neutrons formula (36) is no longer valid, since the total charge is zero. In this case we shall limit ourselves to the first Born approximation which at the considered energy interval should

be valid at small angles; of course one cannot distinguish the scattering of positive particles from that of negative ones, unless one goes over to the second order approximation. The potential is given by

$$U_n(r) = \frac{2\gamma Eeq}{\hbar^2 c^2} \sum_m b_m \exp\left[-\gamma \frac{r}{2}\right] L_m(\gamma r) .$$

One must than evaluate integrals of the type

$$(38) \qquad 2\pi \int_0^\infty r^2 \, dr \, L_n(\gamma r) \exp\left[-\frac{\gamma}{2} r\right] \int_{-1}^{1} dz \exp\left[2ikrz \sin\frac{\theta}{2}\right] =$$

$$= -\frac{4\pi}{\gamma \, \Delta k} \frac{\partial}{\partial \, \Delta k} \left\{ 2 \, \mathrm{Re} \int_0^\infty L_n(2y) \exp[-y] \exp\left[-2i \frac{\Delta k}{\gamma} y\right] dy \right\},$$

where

$$(39) \qquad \Delta k = 2k \sin\frac{\theta}{2} .$$

One could obtain directly the Born approximation for the neutron choosing in eq. (37) $\lambda = 0$ since $\lim F(-i\lambda, 1; \varrho) = 1$. Yet the formula so derived would be much less compact than the corresponding one resulting from the evaluation of the integrals eq. (38). The scattering amplitude is thus [9]

$$(40) \qquad f_n(\theta) = \frac{4Eeq}{\hbar^2 c^2 \, \Delta k} \frac{\partial}{\partial \, \Delta k} \mathrm{Re} \left\{ \sum_m b_m \int_0^\infty \exp[-y] L_m(2y) \exp\left[-2i \frac{\Delta k}{\gamma} y\right] dy \right\} =$$

$$= \frac{4Eeq}{\hbar^2 c^2 \, \Delta k} \frac{\partial}{\partial \, \Delta k} \mathrm{Re} \left\{ \frac{1}{1 + 2i(\Delta k/\gamma)} \sum_m b_m \left(\frac{2i(\Delta k/\gamma) - 1}{2i(\Delta k/\gamma) + 1} \right)^m \right\} .$$

4. – Concluding remarks.

The method which has been used here belongs to the usual phenomenological approach in the problem of reproducing an unknown charge distribution. Yet we have here the freedom of not restricting *a priori* the shape of the potential, since the coefficients of (36) and (40) must be determined from the experimental cross-sections. An eventual magnetic moment distribution as it is necessary in some cases, *e.g.* the neutron, presents no difficulty since it is possible to extend our procedure in a straightforward manner.

* * *

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RIASSUNTO

Si è calcolata l'ampiezza d'urto per la diffusione di particelle cariche da parte di una distribuzione di carica statica e a simmetria sferica. Si suppone che il potenziale corrispondente sia una serie arbitraria di funzioni di Laguerre. Il calcolo è stato fatto usando come approssimazione del primo ordine la funzione d'onda corrispondente alla diffusione da parte della carica puntiforme, e iterando una volta per tener conto dell'estensione della carica.

The Analytic Properties of Scattering Amplitudes in Sixth-Order Perturbation Theory.

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Summary. — The analyticity properties of the sixth-order contribution to the nucleon-nucleon scattering amplitude, as a function of two complex invariants, are investigated. The regularity domain in the first sheet of these invariants is obtained. This domain turns out to be identical with that implied by Mandelstam's representation.

1. – Introduction.

S. Mandelstam [1] conjectured an integral representation for scattering amplitudes $T(s, t)$ in relativistic quantum field theory. The simplest, « non-subtracted » version of this representation has the form:

$$T(s,t) = \frac{1}{\pi^2}\int_{s_0}^{\infty} ds' \int_{t_0}^{\infty} dt' \frac{A_1(s',t')}{(s'-s)(t'-t)} + \frac{1}{\pi^2}\int_{s_0}^{\infty} ds' \int_{u_0}^{\infty} du' \frac{A_2(s',u')}{(s'-s)(u'-u)} + \tag{1.1}$$

$$+ \frac{1}{\pi^2}\int_{t_0}^{\infty} dt' \int_{u_0}^{\infty} du' \frac{A_3(t',u')}{(t'-t)(u'-u)}.$$

s is the square of the energy of the scattered particles in their C.M. system, t is minus the square of their momentum transfer and $u = 2(m_1^2+m_2^2) - s - t$, if m_1 and m_2 are the masses of the particles.

Equation (1.1) implies specific analyticity properties of the scattering amplitude $T(s, t)$, when both variables s and t are allowed to be complex. One

[1] S. Mandelstam: *Phys. Rev.*, **112**, 1344 (1958).

does not know, until now, if these properties are consequence of the general principles of field theory, although S. MANDELSTAM (2) has recently succeeded in deriving from them some weaker properties.

Therefore, it would be interesting to know if a Mandelstam representation exists for the perturbation expansion of the scattering amplitude $T(s, t)$. It has been established that the representation is valid for the fourth-order terms of the perturbation expansion (3), as long as the masses of the scattered and virtual particles satisfy certain inequalities. The representation has also been proved (4) for those terms of the nucleon-nucleon scattering amplitude which correspond to ladder graphs. Furthermore some general properties of the perturbation expansion are known, which are compatible with the Mandelstam representation (5).

In the present paper, we study the sixth-order term of the nucleon-nucleon scattering amplitude, in the case of a Yukawa-type coupling with the meson field. The interest of this investigation is that in the sixth-order approximation appear the first diagrams which give contributions to all the three terms of the right-hand-side of equation (1.1). In all the cases examined until now, each graph contributes to one of these terms only. There is therefore a qualitative difference between sixth-order diagrams and fourth-order or ladder-graphs. We shall concentrate ourselves exclusively on the regularity domain in the first sheet of the sixth-order term. Our result will be a domain which is identical with that implied by the representation (1.1) (6). However, as we ignore the behaviour of $T^{(6)}(s, t)$ at infinity, we cannot write down a definite Mandelstam representation.

Before we examine separately each sixth-order diagram, we indicate the general method we shall use in those cases where the analyticity properties of the diagram under consideration cannot be deduced directly from known results. The problem will be to find out the properties of a function $F(x_1, x_2)$ when both variables x_1 and x_2 take on complex values. $F(x_1, x_2)$ is defined by:

$$F(x_1, x_2) = \lim_{\varepsilon \to 0+} F_\varepsilon(x_1, x_2)\,, \tag{1.2}$$

(2) S. MANDELSTAM: *Nuovo Cimento*, **15**, 658 (1960).

(3) S. MANDELSTAM: *Phys. Rev.*, **115**, 1741 (1959); J. TARSKI: *Journ. Math. Phys.*, **1**, 154 (1960).

(4) G. WANDERS: *Nuovo Cimento*, **17**, 535 (1960); R. CUTKOWSKI: private communication.

(5) J. C. POLKINGHORNE and G. R. SCREATON: *Nuovo Cimento*, **15**, 289, 925 (1960); R. J. EDEN: *Phys. Rev.*, **119**, 1763 (1960); *The problem of proving the Mandelstam representation*, UCRL-9254 (June 1, 1960).

(6) R. J. EDEN, C. ENZ and J. LASCOUX obtained some results on sixth-order diagrams which make the existence of a Mandelstam representation quite plausible (private communication).

where $F_\varepsilon(x_1, x_2)$ has a representation of the following type:

$$(1.3)\quad F_\varepsilon(x_1, x_2) = \int \prod_{i=1}^{5} \mathrm{d}\bar{a}_i \int_0^\infty \mathrm{d}\bar{b}\, \bar{q}(\bar{a}_i, \bar{b}) \cdot [x_1 \bar{f}(\bar{a}_i) + x_2 \bar{g}(\bar{a}_i)\bar{b}^2 - (1 - i\varepsilon)\, \bar{M}^2(\bar{a}_i, \bar{b})]^{-3}.$$

The functions $\bar{f}(\bar{a}\,)$, $g(\bar{a}_i)$ and $\bar{M}^2(\bar{a}_i, \bar{b})$ are real and positive. Equations (1.2) and (1.3) show that for negative values of x_2, $F(x_1, x_2)$ is the boundary value of a function $\tilde{F}(z_1, x_2)$. This function is regular in the z_1-plane, cut along the positive real axis, from x_{10} to infinity, with:

$$(1.4)\quad x_{10} = \mathrm{Min}\, \frac{\bar{M}^2(\bar{a}_i, \bar{b})}{\bar{f}(\bar{a}_i)}.$$

Therefore, one has:

$$(1.5)\quad F(x_1, x_2) = \lim_{\eta \to 0+} \tilde{F}(x_1 + i\eta, x_2) \qquad \text{for } x_2 < 0$$

and one can write:

$$(1.6)\quad \tilde{F}(z_1, x_2) = \sum_{n=0}^{N-1} \frac{1}{n!} c_n(x_2)(x_1 - \xi)^n + \frac{1}{\pi} \int_{x_0}^\infty \mathrm{d}x_1' \frac{(x_1 - \xi)^N \,\mathrm{Im}\, F(x_1', x_2)}{(x_1' - \xi)^N (x_1' - z_1)},$$

with $\xi < x_{10}$, and

$$(1.7)\quad c_n(x_2) = \frac{\partial^n F(x_1, x_2)}{(\partial x_1)^n}\Big|_{x_1 = \xi}.$$

The minimum value one has to give to N in (1.6) results from the behaviour of $\mathrm{Im}\, F(x_1, x_2)$, as x_1 goes to infinity.

The possibility of continuing $\tilde{F}(z_1, x_2)$ to complex values of x_2 depends now, according to (1.6), on the properties of $c_n(x_2)$ and $\mathrm{Im}\, F(x_1, x_2)$ as functions of x_2. As $\xi < x_{10}$, it follows immediately from (1.7) and (1.3) that $c_n(x_2)$ is the boundary value of a function $\tilde{c}_n(z_2)$:

$$(1.8)\quad c_n(x_2) \lim_{\eta \to 0+} \tilde{c}_n(x_2 + i\eta).$$

The regularity domain of $\tilde{c}_n(z_2)$ is the z_2-plane, provided with the cut $(x_{20}, +\infty)$, along the positive real axis:

$$(1.9)\quad x_{20} = \mathrm{Min}\, \frac{\bar{M}^2(\bar{a}_i, \bar{b})}{\bar{g}(\bar{a}_i)\, \bar{b}^2}.$$

In order to get information on $\operatorname{Im} F(x_1, x_2)$, as a function of x_2, we return to $F_\varepsilon(x_1, x_2)$. For arbitrary real values of x_1, the integrand in (1.3) is regular and bounded for complex values $z_2 = x_2 + iy_2$ of x_2 such that:

$$y_2 \bar{g}(\bar{a}_i) \bar{b}^2 + \varepsilon \bar{M}^2(\bar{a}_i, \bar{b}) > 0 ,$$

This condition is satisfied for every possible value of the parameters $\bar{a}_i$ and $\bar{b}$, if

$$y_2 > -\varepsilon x_{20} . \tag{1.10}$$

From this one concludes that (1.3) defines a function $\tilde{F}_\varepsilon(x_1, z_2)$, which is regular in the half-plane D_1: $y_2 > -\varepsilon x_{20}$, with

$$F_\varepsilon(x_1, x_2) = \tilde{F}_\varepsilon(x_1, x_2) \qquad \text{for } x_2 \text{ real.} \tag{1.11}$$

It is impossible to get an enlargement of the regularity domain D_1 of $\tilde{F}_\varepsilon(x_1, z_2)$ without further assumptions on the functions appearing in the representation (1.3). Such an assumption is to suppose that we can perform a change of variables in (1.3), eliminating $\bar{a}_i$ in favour of a_i:

$$\bar{a}_i = \bar{a}_i(a_j, \bar{b}) , \tag{1.1}$$

with the following properties:

a) The functions appearing in the integrand of (1.3) are transformed as:

$$\left| \begin{array}{l} \bar{f}(\bar{a}_i) = f(a_i); \qquad \bar{g}(\bar{a}_i) = g(a_i) , \\ \bar{M}^2(\bar{a}_i, \bar{b}) = x_{20} g(a_i) \bar{b}^2 + M^2(a_i, \bar{b}) , \\ \dfrac{\partial(\bar{a}_1, \ldots, \bar{a}_n)}{\partial(a_1, \ldots, a_n)} \bar{q}(\bar{a}_i, \bar{b}) = q(a_i, \bar{b}) . \end{array} \right. \tag{1.13}$$

b) The function $M^2(a_i, \bar{b})$ is regular and has a negative imaginary part when $\operatorname{Im} \bar{b}^2 < 0$. More precisely:

$$0 > \arg M^2(a_i, \bar{b}) > \arg \bar{b}^2 \quad \text{when} \quad 0 > \arg b^2 > -\pi . \tag{1.14}$$

c) The function $q(a_i, \bar{b})$ iregular when $\operatorname{Im} \bar{b}^2 < 0$.

d) The limits of integration for the new variables a_i are independent of $\bar{b}$.

Now, if we consider $\tilde{F}_\varepsilon(x_1, z_2)$ on the half straight line $z_2 = -w(1 - i\varepsilon)$, ($w$ real and positive), which lies in the regularity domain (1.10), we can further

eliminate $\bar{b}$ by the change of variable:

$$\bar{b}^2 = \frac{1}{w + x_{20}} b^2 , \tag{1.15}$$

and get

$$\tilde{F}_\varepsilon(x_1, -w(1 - i\varepsilon)) = \int \prod_{i=1}^{5} \mathrm{d}a_i \int_0^\infty \mathrm{d}b \, q\left(a_i, \frac{b}{\sqrt{w + x_{20}}}\right) \frac{1}{\sqrt{w + x_{20}}} \cdot \tag{1.16}$$

$$\cdot \left[x_1 f(a_i) - (1 - i\varepsilon) b^2 g(a_i) - (1 - i\varepsilon) M^2 \left(a, \frac{1}{\sqrt{w + x_{20}}}\right)\right]^{-3} .$$

From (1.14) we see that the integrand in (1.16) is regular for

$$0 > \arg (w + x_{20}) > - \arg (1 + i\varepsilon) . \tag{1.17}$$

Thus, returning to the variable z_2, we conclude that $\tilde{F}_\varepsilon(x_1, z_2)$ is regular in the domain D_2:

$$D_2 ; \quad \pi - \arg (1 + i\varepsilon) < \arg (z_2 - (1 - i\varepsilon) x_{20}) < 2\pi - 2 \arg (1 + i\varepsilon) . \tag{1.18}$$

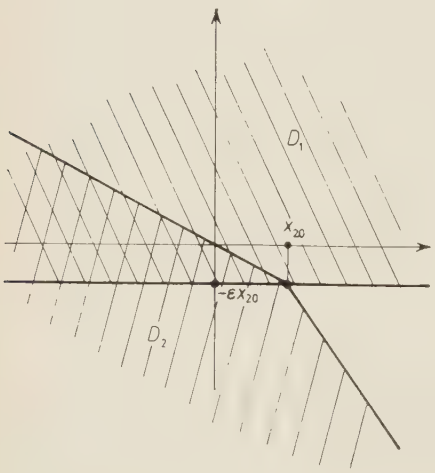

Fig. 1.

As we know already that $\tilde{F}_\varepsilon(x_1, z_2)$ is regular in the half plane D_1, we have obtained for this function the regularity domain $D = D_1 + D_2$ (see Fig. 1). Therefore, according to its definition (1.2), $F(x_1, x_2)$ is the boundary value of a function $\tilde{F}_1(x_1, z_2)$, regular in the z_2-plane, with a cut along the positive real axis, from x_{20} to infinity (7). This implies that $\mathrm{Im}\, F(x_1, x_2)$ can also be continued, from negative values of x_2, into a function which is regular in the cut z_2-plane too.

From this last result, and from our discussion of $c_n(x_2)$, we conclude that

(7) Our discussion is somewhat incomplete in the sense that we tacitely made the usual assumption that integrals like (1.3) and (1.16) converge for all complex values of x_1, resp. w, for which the integrand is regular and bounded. In fact, this point should have to be verified in all the applications of Sect. **2** and Sect. **3**. However, we have no reason to doubt about the convergence of the integrals we shall encounter. Therefore, the rather intricate calculations a complete proof of convergence would involve have not been done in detail.

(1.6) defines an analytic continuation of $\tilde{F}(z_1, x_2)$ into the z_2-plane, provided with cut $(x_{20}, +\infty)$.

To summarize: we have shown that, if there exists a change of variables (1.12), with the properties *a*), *b*), *c*) and *d*), the function $F(x_1, x_2)$, defined by (1.2) and (1.3) is the boundary value of a function $\tilde{F}(z_1, z_2)$:

$$F(x_1, x_2) = \lim_{\substack{\eta_1 \to 0+ \\ \eta_2 \to 0+}} \tilde{F}(x_1 + i\eta_1, x_2 + i\eta_2) \,. \tag{1.19}$$

$\tilde{F}(z_1, z_2)$ is regular in the topological product of the cut z_1-plane with the cut z_2-plane, the cuts lying on the positive real axis and starting at x_{10}, resp. x_{20}. It is convenient to introduce a short-hand notation for these results. We write:

$$F(x_1, x_2) \text{ is b.v. of a fct. reg. in } 0 < \arg(x_1 - x_{10}) < 2\pi \,, \quad 0 < \arg(x_2 - x_{20}) < 2\pi \,,$$

where « b.v. » stands for « boundary value », « fct. » for « function » and « reg. » for « regular ».

In the following sections we examine successively all sixth-order diagrams. For some of these graphs the domain of regularity follows directly from available results. For the remaining ones, we have to set up a representation of the form (1.3), and to prove the existence of a change of variable (1.12) satisfying the conditions *a*) to *d*).

2. – Diagrams with one or two exchanged mesons.

Let us first define our notations: p_1 and p_2 are the four-momenta of the incoming nucleons, p_1' and p_2' those of the scattered nucleons. We introduce the vectors:

$$\begin{cases} P = p_1 + p_2 = p_1' + p_2' \,, \\ k = (p_1 - p_2)/2 \,, \quad k' = (p_1' - p_2')/2 \,. \end{cases} \tag{2.1}$$

The invariants s, t and u are then given by

$$\begin{cases} s = -P^2 = W^2 = 4(m^2 + k^2) = 4(m^2 + k'^2) \,, \\ t = -4\Delta^2 = -(k - k')^2 = -2k^2(1 - \cos\theta) \,, \\ u = -(k + k')^2 = -2k^2(1 + \cos\theta) \,. \end{cases} \tag{2.2}$$

One has the linear relation

$$s + t + u = 4m^2 . \tag{2.3}$$

In this section, we consider those sixth-order diagrams in which a net exchange of one or two mesons takes place between the nucleons. They are generated from fourth-order diagrams by insertion of self-energy- or vertex-parts. Typical diagrams are indicated in Fig. 2 and Fig. 3; all the other ones are obtained from these diagrams by permutations between incoming and outgoing nucleons. These permutations are equivalent to substitutions between the variables s, t and u, in the functions $T_i(s, t, u)$ corresponding to the graphs of Fig. 2 and Fig. 3.

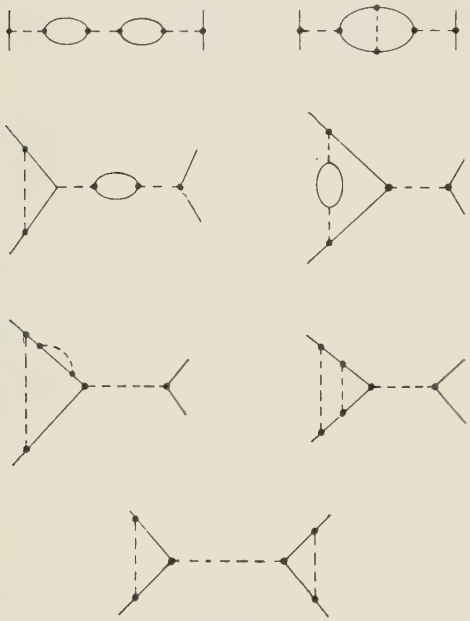

Fig. 2.

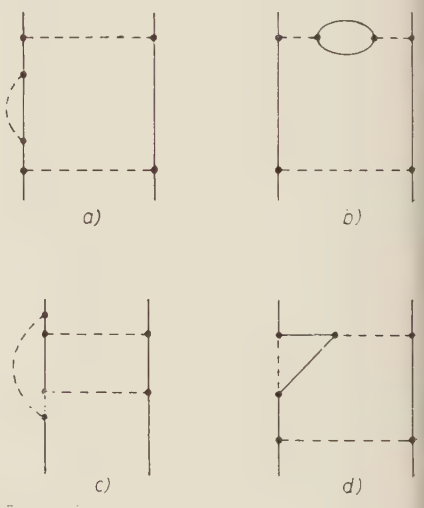

Fig. 3.

The diagrams of Fig. 2 lead to functions $T_i(t)$ of t alone. From the known analyticity properties of renormalized self-energies and form-factors ([8]), it follows in an obvious way that these functions are analytic functions of t. They have a pole at $t = \mu^2$ (μ = meson mass), and a cut along the positive real axis, starting at $t = 4\ m^2$:

A) $T_i(t)$ is b.v. of a fct. reg. in $0 < \arg(t - 4m^2) < 2\pi$, except a pole at $t = \mu^2$.

([8]) See for example: K. SYMANZIK: *Progr. Theor. Phys.*, **20**, 690 (1958).

The insertion of a self-energy-part into an inner line substitutes a superposition of propagators with higher masses to the propagator initially associated with that line. Thus, the proof of the Mandelstam representation for fourth-order graphs ([3]) also apply to the diagrams a and b of Fig. 3. This implies that:

B) $T_a(s, t)$ and $T_b(s, t)$ are b.v. of fcts. reg. in $0 < \arg(s - 4m^2) < 2\pi$, $0 < \arg(t - 4\mu^2) < 2\pi$.

Diagram c in Fig. 3 is of the ladder-type, if seen from the right to the left, and graphs of this type have been examined in ([4]). Straightforward application of the method displayed in this work shows that a Mandelstam representation holds for $T_c(s, t)$. Thus:

C) $T_c(s, t)$ is b.v. of a fct. reg. in $0 < \arg(s - (2m + \mu)^2) < 2\pi$, $0 < \arg(t - 4\mu^2) < 2\pi$.

The vertex-part of diagram d in Fig. 3 has two exterior momenta off shell; therefore, known results about form-factors are of no help here, and a detailed treatment is required. We shall apply the method exposed in Section **1**. Our first task is to build up a representation of the form (1.3) for $T_d(s, t)$. To this end we use the following representations of the nucleon- and meson-propagators ([9]):

$$D_F(p, m) = \lim_{\varepsilon \to 0+} \frac{1}{p^2 + (1 - i\varepsilon)m^2}; \qquad D_F(p, \mu) = \lim_{\varepsilon \to 0+} \frac{1}{p^2 + (1 - i\varepsilon)\mu^2}. \tag{2.4}$$

Products of propagators are parametrized according to

$$\frac{1}{a^n b^m} = \frac{(n + m - 1)!}{(n - 1)!\,(m - 1)!} \int_0^\infty \mathrm{d}x \, \frac{x^{m-1}}{(a + bx)^{n+m}}. \qquad \text{if } \operatorname{Im} ab > 0. \tag{2.5}$$

This gives:

$$T_d(s, t) = \text{const} \int_0^\infty \mathrm{d}v \int_0^\infty \mathrm{d}\bar{w} \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \int_0^{\pi/2} \mathrm{d}\vartheta \int_0^\infty \mathrm{d}b \; \bar{q}(v, \bar{w}, x, y, \vartheta, b) \cdot \tag{2.6}$$

$$\cdot [t f(v, y) + s g(x, y) b^2 - (1 - i\varepsilon)\, \bar{M}^2(v, \bar{w}, x, y, \vartheta, b)]^{-3},$$

([9]) For simplicity, we assume the nucleon and the meson to be scalar particles. As is well-known, this simplification does not alter the regularity-domain of the scattering amplitude.

with:

$$
(2.7)\quad \begin{cases} \bar{q}(v,\bar{w},x,y,\theta,b)=\dfrac{b^2\cos\theta}{1+(1+b\cos\theta)\bar{w}}\left[1+v+y+b\big((1+x)\cos\theta+\sin\theta\big)\right], \\ f(v,y)=v(1+y)\,, \\ g(x,\theta)=(1+x)\cos\theta\sin\theta\,, \\ \bar{M}^2(v,\bar{w},x,y,\theta,b)=m^2\big((1+x)\cos\theta+\sin\theta\big)^2b^2+ \\ +\left[2m^2+\mu^2(v+y)+\mu^2b\cos\theta+(m^2+\mu^2b(1+b\cos\theta)\cos\theta)\bar{w}+m^2\frac{1}{\bar{w}}\right]\cdot \\ \cdot\big(1+v+y+b\big((1+x)\cos\theta+\sin\theta\big)\big)\,. \end{cases}
$$

Elementary calculations give the quantities t_0 and s_0, corresponding to x_{10} and x_{20} in Section **1**:

$$
(2.8)\quad \begin{cases} t_0=\operatorname{Min}\bar{M}^2(x,\bar{w},x,y,\theta,b)/f(v,y)=4\mu^2\,, \\ s_0=\operatorname{Min}\bar{M}^2(v,\bar{w},x,y,\theta,b)/g(x,\theta)b^2=4m^2\,. \end{cases}
$$

A change of variable satisfying the conditions *a*) to *d*) of Section **1** is:

$$
[2.9]\qquad \bar{w}=m\left[m^2+\mu^2b(1+b\cos\theta)\cos\theta\right]^{-\frac{1}{2}}w\,,
$$

It leaves the functions $f(v,y)$ and $g(x,\theta)$ unchanged, and

$$
(2.10)\quad \begin{cases} q(v,w,x,y,\theta,b)=\dfrac{\partial\bar{w}}{\partial w}\,\bar{q}(v,\bar{w},x,y,\theta,b)= \\ \qquad=\dfrac{mb^2\left[1+v+y+b\big((1+x)\cos\theta+\sin\theta\big)\right]\cos\theta}{(m^2+\mu^2b(1+b\cos\theta)\cos\theta)^{\frac{1}{2}}+m(1+b\cos\theta)w}\,, \\ M^2(v,w,x,y,\theta,b)=\bar{M}^2(v,\bar{w},x,y,\theta,b)-s_0g(x,\theta)b^2= \\ \qquad=m^2\big((1+x)\cos\theta-\sin\theta\big)^2b^2+\left[2m^2+\mu^2(v+y)+\mu^2b\cos\theta+\right. \\ \qquad+\left.m(m^2+\mu^2b(1+b\cos\theta)\cos\theta)^{\frac{1}{2}}\left(w+\frac{1}{w}\right)\right]\cdot \\ \qquad\cdot\big(1+v+y+b\big((1+x)\cos\theta+\sin\theta\big)\big)\,. \end{cases}
$$

In particular, it is clear that $M^2(v,w,x,y,\theta,b)$ verifies the condition (1.14). Therefore we conclude that:

D) $T_d(s,t)$ is b.v. of a fct. reg. in $0<\arg(s-4m^2)<2\pi$, $0<\arg(t-4\mu^2)<2\pi$.

3. – Diagrams with three exchanged mesons.

The sixth-order diagrams we left out until now are characterized by an effective exchange of three mesons between the nucleons. There are two basic diagrams of this type, which can be chosen as in Fig. 4. All the remaining graph sare obtained from them by suitable permutations.

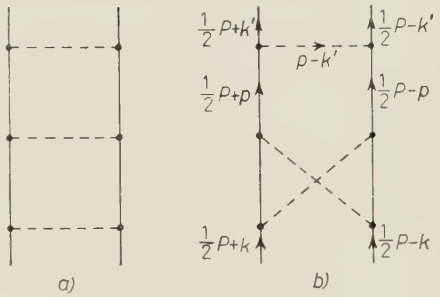

Fig. 4.

Diagram a in Fig. 4 is a special case of the ladder-graphs investigated in [4]. It was shown there that a Mandelstam representation exists for the corresponding scattering amplitude $T_a(s,t)$. This implies that:

E) $T_a(s,t)$ is b.v. of a fct. reg. in $0<\arg(s-4m^2)<2\pi$, $0<\arg(t-9\mu^2)<2\pi$.

Diagram b in Fig. 4 is the crucial diagram of the sixth-order approximation, and we discuss it according to the general method described in Section 1. In order to get a convenient expression for $T_b(s,t)$, we start from the «crossed-part» $K(P,p,k)$ of diagram b; it is given by

$$(3.1)\qquad K(P,p,k)=\text{const}\int_0^\infty dx\int_0^\infty dy\int_0^\infty \frac{dv}{v}\left\{\left[\left(\frac{1}{2}P+p\right)^2+m^2\right]x+\right.$$
$$\left.+\left[\left(\frac{1}{2}P-p\right)^2+m^2\right]y+(p+k)^2xy+(p-k)^2+(1-i\varepsilon)M_0^2(x,y,v)\right\}^{-1}.$$

with

$$(3.2)\qquad M_0^2(x,y,v)=2m^2xy+\mu^2(2+x+y)+$$
$$+(m^2x^2+\mu^2(1+x))v+(m^2y^2+\mu^2(1+y))\frac{1}{v}.$$

From this, one gets:

$$(3.3)\qquad T_b(s,t)=\text{const}\int d^4p\,D_F\left(\frac{1}{2}P+p,m\right)D_F\left(\frac{1}{2}P-p,m\right)D_F(p-k',\mu)\cdot$$
$$\cdot K(P,p,k)=\text{const}\int_0^\infty dx\int_0^\infty dy\int_0^\infty da\int_a^\infty db\int_0^\infty dz\int_0^\infty \frac{dv}{v}\,xy\cdot$$
$$\cdot(1+z+(1+a)x+(1+b)y+xy)\frac{1}{A^3}.$$

There are three equivalent forms for A, obtained by expressing this quantity successively in terms of the three pairs of invariants (s, t), (s, u) and (u, t):

$$(3.4)\quad \begin{cases} A = A_1(s,t) = s(ab+a+b-z)xy + tz(1-xy) - (1-i\varepsilon)\varkappa_1^2, \\ A = A_2(s,u) = s\big(xy(ab+a+b)-z\big) + uz(xy-1) - (1-i\varepsilon)\varkappa_2^2, \\ A = A_3(t,u) = t\big(z-xy(ab+a+b)\big) + u(z-ab-a-b)xy - (1-i\varepsilon)\varkappa_3^2. \end{cases}$$

We remark that in each of these forms of A, the coefficients of the invariants s, t and u fail to be positive for all values of the parameters. This means That $T_b(s, t)$ itself, as given by (3.3) cannot be brought into the form (1.3). However, we note the identity:

$$(3.5)\quad 1 = \theta(ab+a+b-z)\,\theta(1-xy) + \theta\big(xy(ab+a+b)-z\big)\,\theta(xy-1) + \\ + \theta(z-ab-a-b)\,\theta\big(z-xy(ab+a+b)\big)\,.$$

The right-hand-side of this identity can be freely introduced into the integrand of (3.3); $T_b(s, t)$ is thereby splitted into three terms. In each of these terms the invariants will have positive coefficients if one inserts in it that expression of A in which the coefficients of the invariants coincide with the arguments of the step-functions. One gets in this way:

$$(3.6)\qquad T_b(s,t) = T_1(s,t) + T_2(s,u) + T_3(t,u)\,,$$

with

$$(3.7a)\qquad T_1(s,t) = \text{const}\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}z \int_0^\infty \mathrm{d}a \int_a^\infty \mathrm{d}b \int_0^\infty \frac{\mathrm{d}v}{v}\cdot$$

$$\cdot\,\theta(ab+a+b+z)\,\theta(1-xy)\,xy\big(1+z+(1+a)x+(1+b)y+xy\big)\,\frac{1}{[A(s,t)]^3}\,,$$

$$(3.7b)\qquad T_2(s,u) = \text{const}\int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \int_0^\infty \mathrm{d}z \int_0^\infty \mathrm{d}a \int_a^\infty \mathrm{d}b\,\frac{\mathrm{d}v}{v}\cdot$$

$$\cdot\,\theta\big(xy(ab+a+b)-z\big)\,\theta(xy-1)\,xy\big(1+z+(1+a)x+(1+b)y+xy\big)\,\frac{1}{[A(s,u)]^3}\,,$$

$$(3.7c)\qquad T_3(t,u) = \ldots\,.$$

In the following we show that the functions $T_1(s, t)$, $T_2(s, u)$ and $T_3(t, u)$ can be continued into the complex planes of their arguments, provided with suitable cuts.

Analyticity of $T_1(s, t)$. In order to get an expression for $T_1(s, t)$ of the form (1.3), we eliminate x, y, a and b in favour of β, λ, α and ϱ by the following changes of variables:

$$\text{(3.8)} \qquad \begin{cases} xy = \dfrac{\beta}{1+\beta}\,; \quad \dfrac{x}{y} = (1 + (\lambda\beta + z)\alpha)\varrho^2\,, \\ ab + a + b - z = \lambda\beta\,; \quad \dfrac{1+b}{1+a} = 1 + (\lambda\beta + z)\alpha\,. \end{cases}$$

This leads to:

$$\text{(3.9)} \qquad \begin{cases} T_1(s, t) = \text{const} \int_0^\infty \mathrm{d}z \int_0^\infty \mathrm{d}\bar{v} \int_0^\infty \mathrm{d}\varrho \int_0^\infty \mathrm{d}\lambda \int_0^1 \mathrm{d}\alpha \int_0^\infty \mathrm{d}\beta\,, \\ \bar{q}(z, \bar{v}, \varrho, \lambda, \alpha, \beta)[tz + s\lambda\beta^2 - (1 - i\varepsilon)\, \bar{M}_1^2(z, \bar{v}, \varrho, \lambda, \alpha, \beta)]^{-3}\,, \end{cases}$$

where

$$\text{(3.10)} \qquad \begin{cases} \bar{M}_1^2(z, \bar{v}, \varrho, \lambda, \alpha, \beta) = m^2 \left[\left(\varrho + \dfrac{1}{\varrho}\right)^2 \lambda\beta^2 + (1+z)\left(\varrho - \dfrac{1}{\varrho}\right)^2 \beta\right] + \\ \quad + \left[1 + z + \sqrt{\dfrac{\beta(1+z+\beta)}{1+\beta}}\left(\varrho + \dfrac{1}{\varrho}\right) + \dfrac{\beta}{1+\beta}\right]\left[\mu^2(2+z)(1+\beta) + \right. \\ \quad + \mu^2\left(\varrho\sigma + \dfrac{1}{\varrho\sigma}\right)\sqrt{\beta(1+\beta)} + 2m^2\beta + (\mu^2 + \mu^2\varrho\sigma\sqrt{\beta(1+\beta)} + \\ \quad \left. + m^2\varrho^2\sigma^2\beta)\bar{v} + \left(\mu^2 + \mu^2\,\dfrac{\sqrt{\beta(1+\beta)}}{\varrho\sigma} + m^2\,\dfrac{\beta}{\varrho^2\sigma^2}\right)\dfrac{1}{\bar{v}}\right], \\ \bar{q}(z, \bar{v}, \varrho, \lambda, \alpha, \beta) = \dfrac{\beta(\lambda\beta + z)}{\bar{v}\varrho\sigma}\left(1 + z + \sqrt{\dfrac{\beta(1+z+\lambda\beta)}{1+\beta}}\left(\varrho + \dfrac{1}{\varrho}\right) + \dfrac{\beta}{1+\beta}\right). \end{cases}$$

In these equations, the abbreviation:

$$\sigma = [1 + (z + \lambda\beta)\alpha]^{\frac{1}{2}}.$$

has been used. The minima s_0 and t_0 (which are equivalent to the quantities x_{10} and x_{20} of Section 1) are

$$\text{(3.11)} \qquad s_0 = \text{Min}\, \bar{M}_1^2/\lambda\beta^2 = 4m^2\,; \qquad t_0 = \text{Min}\, \bar{M}_1^2/z = 9\mu^2\,.$$

We now claim that the change of variable

$$\text{(3.12)} \qquad \bar{v} = \left[\mu^2 + \mu^2\,\frac{\sqrt{\beta(1+\beta)}}{\varrho\sigma} + m^2\,\frac{\beta}{\varrho^2\sigma^2}\right]^{\frac{1}{2}}\left[\mu^2 + \mu^2\varrho\sigma\sqrt{\beta(1+\beta)} + m^2\varrho^2\sigma^2\beta\right]^{-\frac{1}{2}} v\,,$$

satisfies the conditions *a*), *b*), *c*) and *d*) of Section **1**. This is obvious for conditions *a*) and *d*). Noting that:

$$q(z, v, \varrho, \lambda, \alpha, \beta) = \frac{\partial \bar{v}}{\partial v}\, \bar{q}(z, \bar{v}, \varrho, \lambda, \alpha, \beta) = \bar{q}(z, v, \varrho, \lambda, \alpha, \beta)\,, \tag{3.13}$$

it is clear, from the expression (3.10) of $\bar{q}$, that condition *c*) is also verified. Furthermore:

$$\begin{aligned} M_1^2(z, v, \varrho, \lambda, \alpha, \beta) &= \bar{M}_1^2(z, \bar{v}, \varrho, \lambda, \alpha, \beta) - s_0\lambda\beta^2 = \\ &= m^2\left(\varrho - \frac{1}{\varrho}\right)^2(1 + z + \lambda\beta) + \left[1 + z + \sqrt{\frac{\beta(1 + z + \lambda\beta)}{1 + \beta}} + \frac{\beta}{1 + \beta}\right]\cdot \\ &\cdot\left[\mu^2(2 + z)(1 + \beta) + \mu^2\left(\varrho\sigma + \frac{1}{\varrho\sigma}\right)\sqrt{\beta(1 + \beta)} + 2m^2\beta + \right. \\ &\left. + \left(\left(\mu^2 + \mu^2\varrho\sigma\sqrt{\beta}(1 + \beta) + m^2\varrho^2\sigma^2\beta\right)\left(\mu^2 + \mu^2\,\frac{\sqrt{\beta(1 + \beta)}}{\varrho\sigma} + m^2\,\frac{\beta}{\varrho^2\sigma^2}\right)\right)^{\frac{1}{2}}\left(v + \frac{1}{v}\right)\right]. \end{aligned} \tag{3.14}$$

It is elementary, though somewhat tedious, to investigate the behaviour of M_1^2 as β becomes complex (β plays the role of b, in Section **1**). The result is that M_1^2 satisfies the condition (1.14). According to the discussion in Section **1**, we may therefore conclude that:

F) $T_1(s, t)$ is b.v. of a fct. reg. in $0 < \arg(s - 4m^2) < 2\pi$, $0 < \arg(t - 9\mu^2) < 2\pi$.

Analyticity of $T_2(s, u)$. The elimination of x, y and z from the expression (3.7*b*) of $T_2(s, u)$ by the changes of variables:

$$x = 1/x'\,, \qquad y = 1/y'\,, \qquad z = z'/x'y'\,, \tag{3.15}$$

leads to a new expression which is very close to that of $T_1(s, t)$. One gets it from the expression (3.7*a*) of $T_1(s, t)$ by substituting u to t and by changing at some places m^2 into μ^2 or μ^2 into m^2. These modifications do not alter the possibility of analytic continuation, but merely displace the starting points of the cuts. After inspection, one finds that:

G) $T_2(s, u)$ is b.v. of a fct. reg. in $0 < \arg(s - 4m^2) < 2\pi$, $0 < \arg(u - (2m + \mu)^2) < 2\pi$.

Analyticity of $T_3(t, u)$. The following changes of variables bring $T_3(u, t)$ into the form (1.3):

$$\text{(3.16)} \qquad \begin{cases} xy = \dfrac{(1+\beta)\eta\beta}{1+\eta\beta}; & \dfrac{x}{y} = \operatorname{tg}\varphi\,, \\ ab + a + b = \bar{\lambda}; & \dfrac{a}{b} = \operatorname{tg}\theta\,, \\ z = (1+\beta)\bar{\lambda}\,. \end{cases}$$

One gets:

$$\text{(3.17)} \qquad \begin{cases} T_3(t, u) = \text{const} \int_0^\infty \mathrm{d}\eta \int_0^\infty \mathrm{d}v \int_0^\infty \mathrm{d}\bar{\lambda} \int_{\pi/4}^{\pi/2} \mathrm{d} \int_0^{\pi/2} \mathrm{d}\varphi \int_0^\infty \mathrm{d}\beta\,, \\ \bar{q}(\eta, v, \varphi, \, , \bar{\lambda}, \beta)[t + u\eta\beta^2 - (1 - i\varepsilon)\, \bar{M}_3^2(\eta, v, \varphi, \theta, \bar{\lambda}, \beta)]^{-3}\,, \end{cases}$$

where

$$\text{(3.18)} \qquad \begin{cases} \bar{M}_3^2(\eta, v, \varphi, \theta, \bar{\lambda}, \beta) = A_1 + A_2\eta\beta + A_3\eta\beta^2 + \\ + A_4\sqrt{(1+\beta)(1+\eta\beta)\eta\beta} + \bar{\lambda}\mu^2(1+\beta)(1+\eta\beta) + \\ + \frac{1}{\bar{\lambda}}\frac{1}{1+\beta}\Big[(1+\eta\beta) + \frac{1}{\sin\varphi\cos\varphi}\sqrt{(1+\beta)(1+\eta\beta)\eta\beta} + (1+\beta)\eta\beta\Big]\cdot \\ \cdot\Big[B_1 + B_2\sqrt{\frac{(1+\beta)\eta\beta}{1+\eta\beta}} + B_3\frac{(1+\beta)\eta\beta}{1+\eta\beta}\Big] + \\ + \frac{1}{c + (c^2 + \bar{\lambda})^{\frac{1}{2}}}\Big[C_1\Big(B_1 + B_2\sqrt{\frac{(1+\beta)\eta\beta}{1+\eta\beta}} + \frac{(1+\beta)\eta\beta}{1+\eta\beta}B_3\Big)\cdot \\ \cdot\sqrt{\frac{(1+\eta\beta)\eta\beta}{1+\beta}} + C_2\eta\beta + C_3\lambda\sqrt{(1+\beta)(1+\eta\beta)}\,\eta\beta\Big] + D\,\Big[c + \frac{\lambda\eta\beta}{(c^2+\lambda)^{\frac{1}{2}}}\Big]^2\,, \\ \\ \bar{q}(\eta, v, \varphi, \, , \bar{\lambda}, \beta) = E\,\frac{1}{\bar{\lambda}^2}\,\frac{\eta\beta^2}{1+\beta}\Big[1 - \frac{E_1}{(c^2+\bar{\lambda})^{\frac{1}{2}}}\Big]\cdot \\ \cdot\Big[1 + (1+\beta)\lambda + \frac{(1+\beta)\eta\beta}{1+\eta\beta} + \Big(E_2 + E_3\,\frac{\lambda}{c + (c^2+\bar{\lambda})^{\frac{1}{2}}}\Big)\cdot\sqrt{\frac{(1+\beta)\eta\beta}{1+\eta\beta}}\Big]\cdot \end{cases}$$

The quantities A_j, B_j, C_j, D, E, E_j, and c are positive functions of v, φ and θ, which have not to be specified. A change of variable realizing the conditions *a*), *b*), *c*) and *d*) of Section **1** is:

$$\text{(3.19)} \qquad \bar{\lambda} = \frac{1}{\mu^2(1+\beta)}\Big[B_1 + B_2\sqrt{\frac{(1+\beta)\eta\beta}{1+\eta\beta}} + B_3\frac{(1+\beta)\eta\beta}{1+\eta\beta}\Big]\lambda\,.$$

It is easy to see that the conditions *a*), *c*) and *d*) are verified. To show that $(\bar{M}_3^2 - (2m+\mu)^2 \eta\beta^2)$ is transformed into a function M_3^2 satisfying condition *b*) requires some lengthy, but elementary, calculations, which we shall not reproduce here. Our conclusion is:

H) $T_3(t, u)$ is b.v. of a fct. reg. in $0 < \arg(t - 9\mu^2) < 2\pi$, $0 < \arg(u - (2m+\mu)^2) < 2\pi$.

Collecting our results *A*) to *H*), we see that the sixth-order contribution $T^{(6)}(s, t)$ to the nucleon-nucleon scattering amplitude is a sum of five terms:

$$T^{(6)}(s, t) = \frac{\text{const}}{t+\mu^2} + T_0^{(6)}(t) + T_1^{(6)}(s, t) + T_2^{(6)}(s, u) + T_3^{(6)}(t, u) . \tag{3.20}$$

The last four terms can be continued as analytic functions into the complex planes of their arguments, provided with suitable cuts. As is easily checked, the starting points of these cuts are identical with the normal thresholds one obtains from the well-known heuristic rules for constructing a Mandelstam representation. Therefore, a representation of the type (1.1), or a « subtracted version » of (1.1), is valid for $T^{(6)}(s, t)$, if it is bounded by a polynomial in s and t at infinity. The method used in this paper gives no information on the detailed structure of the functions $A_1(s, t)$, $A_2(s, u)$ and $A_3(t, u)$ in (1.1). In particular, we cannot prove that $A_1(s, t)$, for example, differs from zero only in some region $t > t_0(s)$ $(t_0(\infty) = t_0,\ t_0(s_0) = \infty)$. This structure could be determined by the methods developed by J. C. Polkinghorne and F. R. Screaton ([6]) and R. J. Eden and collaborators ([5,6]). The main purpose of the present paper was to prove that $T^{(6)}(s, t)$ has no singularities for complex and finite values of s and t.

RIASSUNTO (*)

Si esamina il dominio di regolarità delle contribuzioni di sesto ordine all'ampiezza di scattering nucleone-nucleone. Risulta che questo dominio è identico a quello implicato dalla rappresentazione di Mandelstam.

(*) *Traduzione a cura della Redazione.*

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Anomalous Thresholds in Dispersion Theory - I (*).

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Summary. — The form factor of a particle in the so-called anomalous case (loosely bound system) is studied from the viewpoint of: 1) description in terms of a Schrödinger-type wave function; 2) perturbation theory; and 3) dispersion theory. A prescription is given on how to calculate the absorptive part of a dispersion relation in the correct Riemann sheet.

1. – Introduction.

It is our purpose here to present a detailed discussion of anomalous thresholds ([1,2]) in form factors and partial wave scattering amplitudes from the point of view of dispersion theory. In a later paper this discussion will be extended to the full scattering amplitude as a function of both the energy and momentum transfer variables. We will restrict ourselves to two-particle intermediate states, certainly a reasonable restriction from the computational point of view. The procedure presented here is probably extendable to more general intermediate states with very little change, as will be indicated. It will be shown with the aid of perturbation theory that a straightforward ap-

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(1) Y. Nambu: *Nuovo Cimento*, **9**, 610 (1958).

(2) R. Karplus, C. M. Sommerfeld and E. H. Wichmann: *Phys. Rev.*, **111**, 1187 (1958); **114**, 376 (1959).

plication of the calculational rules of dispersion theory leads one to the correct analytic function but in the presence of anomalous thresholds one is on the wrong Riemann sheet in the momentum transfer variable. A simple procedure is presented for performing the continuation to the physical sheet, thus making it possible to use the powerful techniques of dispersion theory in these anomalous cases. It will be shown that the presence of anomalous thresholds actually simplifies the calculation of form factors, in that much of the contribution to the form factor in this case is of purely kinematical origin and therefore perhaps one does not need to understand the dynamics as well as in the normal threshold case.

In Section **2** the form factor is discussed from the Schrödinger equation point of view in order to aid in the physical interpretation of our later presentation. This example is presented in a slightly novel manner so as to simplify the comparison with the field theoretic discussion. In Section **3** a discussion of the form factor is carried out in lowest order perturbation theory and compared to the non-relativistic example. In Section **4** this discussion is repeated from the dispersion theoretic point of view and it is shown that an analytic continuation of the conventional result to another Riemann sheet yields the same result as perturbation theory, and thus in the non-relativistic limit will agree with the Schrödinger equation form factor. In the course of this development some rescattering corrections are included by considering the partial wave scattering amplitude in the presence of anomalous thresholds. Finally, in Section **5**, some last remarks and extensions are made.

2. – Schrödinger theory.

It is clear that the type of potential which yields results in closest accord with the analytical properties of field theory is the Yukawa or a linear superposition thereof. Perhaps the most relevant discussions of this point are the proofs of the two dimensional Mandelstam representation for the scattering amplitudes ([3]). Also of interest in this connection is a discussion of the connection between the dispersion relation for certain vertex functions and the Schrödinger equation ([4]). These presentations rest crucially on the assumption that the potential is expressible in the form

$$rV(r) = \int_{\mu}^{\infty} \mathrm{d}m\, v(m) \exp\left[-mr\right]. \tag{2.1}$$

([3]) R. BLANKENBECLER, M. GOLDBERGER, N. KHURI and S. TREIMAN: *Ann. Phys.* (in press).

([4]) R. BLANKENBECLER and L. COOK: submitted to *Phys. Rev.*

With this type of potential, it is simple to show that the wave function for the two particle bound S-state (which will be occasionally referred to as the deuteron) is expressible in the form [4]

$$\varphi(x) = \int_{(\mu+\alpha^2)}^{\infty} \mathrm{d}x' \sigma(x')^2 (x' - \alpha^2)[(x+\alpha^2)(x'+x)]^{-1}, \tag{2.2}$$

where $\alpha^2 = M\varepsilon$, ε is the (positive) binding energy, and x is the square of the relative momentum. It is found that $\sigma(x')$ has discontinuities in some derivative at $x' = (N\mu+\alpha)^2$, $N = 2, 3, \ldots$.

The form factor is

$$F(x) = \int \mathrm{d}^3 p \varphi[(\boldsymbol{p}-\boldsymbol{q})^2]\varphi[\boldsymbol{p}^2], \tag{2.3}$$

where $s = -4\boldsymbol{q}^2$ is the momentum transfer variable. This can be expressed as

$$F(s) = N^2 I(s; \alpha^2, \alpha^2) - 2N \int_{(\mu+\alpha)^2}^{\infty} \mathrm{d}x \sigma(x) I(s; x, \alpha^2) + \int_{(\mu+\alpha)^2}^{\infty} \mathrm{d}x\, \mathrm{d}y \sigma(x)\sigma(y) I(s; x, y), \tag{2.4}$$

where

$$I(s; t, u) = \int \mathrm{d}^3 p [(\boldsymbol{p}-\boldsymbol{q})^2 + t]^{-1} [\boldsymbol{p}^2 + u]^{-1}, \tag{2.5}$$

and

$$N = \int_{(\mu+\alpha)^2}^{\infty} \mathrm{d}x \sigma(x).$$

The integral I is easily evaluated by combining denominators and one finds

$$I(s; t, u) = 2\pi \int_0^1 \mathrm{d}z \int_0^{\infty} \mathrm{d}p [p^2 - \tfrac{1}{4}sz(1-z) + zt + (1-z)u]^{-1}. \tag{2.6}$$

Thus it is seen that $I(s)$ is an analytic function with a cut for positive s, and can be expressed in standard dispersion form. It is easily seen that

$$\operatorname{Im} I(s; t, u) = \frac{2\pi^2}{\sqrt{s}} \theta[s - 4(\sqrt{t} + \sqrt{u})^2]. \tag{2.7}$$

Combining these results, we see that the form factor is defined by the statement

$$F(s) = \frac{1}{\pi} \int_{16\alpha^2}^{\infty} \mathrm{d}s' \, \frac{\mathrm{Im}\, F(s')}{s' - s - i\varepsilon}, \tag{2.8}$$

where

$$\frac{\sqrt{s}}{2\pi^2} \mathrm{Im}\, F(s) = N^2 \theta(s - 16\alpha^2) - 2N \int_{(\mu+\alpha)^2}^{\infty} \mathrm{d}x \, \sigma(x) \theta[s - 4(\sqrt{x} + \alpha)^2] + \tag{2.9}$$

$$+ \int_{(\mu+\alpha)^2}^{\infty} \mathrm{d}x \, \mathrm{d}y \, \sigma(x) \sigma(y) \theta[s - 4(\sqrt{x} + \sqrt{y})^2].$$

A very similar absorptive part will be found in the field theoretic case and will then suggest a physical interpretation of an anomalous threshold. This point has been discussed by OEHME ([5]). For a very thorough investigation of this point, see ref. ([4]). We turn now to a discussion of the form factor in perturbation theory.

3. – Perturbation theory.

A short review of the well discussed ([1-3]) lowest order vertex will be presented. The model which we choose to study is composed of scalar deuterons and nucleons interacting with an external scalar photon. All particles are assumed to have isotopic spin zero and all meson effects will be neglected for simplicity. The diagram of interest is shown in Fig. 1.

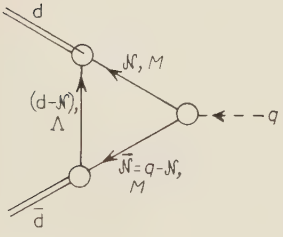

Fig. 1.

The particle exchange in the $\mathcal{N} + \overline{\mathcal{N}} \to \mathrm{d} + \bar{\mathrm{d}}$ scattering is assumed to have mass Λ^2 for generality. Our primary aim, of course, is to examine the analytic structure of the form factor and to discuss in detail its absorptive part. This has been thoroughly discussed by OEHME ([5]), but we will give a brief summary in the interest of completeness.

([5]) R. OEHME: *Nuovo Cimento*, **13**, 778 (1959).

First, define $q^2 = (d+\bar{d})^2 = -s$, then the form factor in terms of parametric integrals is

$$(3.1)\quad F(s, \Lambda^2) = \\ = \frac{\Gamma_0^2}{4\pi}\int_0^1 \mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\ \delta(\alpha-\beta-\gamma)[\Lambda^2(1-\alpha)+M^2\alpha-M_\mathrm{d}^2\alpha(1-\alpha)-s\beta(\alpha-\beta)]^{-1}.$$

If the integration variables are changed so that $F(s, \Lambda^2)$ achieves a dispersion form, the result is

$$(3.2)\quad F(s, \Lambda^2) = \frac{\Gamma_0^2}{\pi}\int_{s_0}^{4M^2}\frac{\mathrm{d}s'}{(s'-s)\sqrt{s'(4M_\mathrm{d}^2-s')}}\int_{\alpha_-}^{\alpha_+}\mathrm{d}\alpha[(\alpha-\alpha_-)(\alpha_+-\alpha)]^{-\frac{1}{2}} + \\ + \frac{\Gamma_0^2}{\pi}\int_{4M^2}^{\infty}\frac{\mathrm{d}s'}{(s'-s)\sqrt{s'(4M_\mathrm{d}^2-s')}}\int_{\alpha_-}^{1}\mathrm{d}\alpha[(\alpha-\alpha_-)(\alpha_+-\alpha)]^{-\frac{1}{2}},$$

where

$$(3.3)\quad s_0 = 4M_\mathrm{d}^2 - (M_\mathrm{d}^2+\Lambda^2-M^2)/\Lambda^2,$$

and

$$(3.4)\quad \alpha_\pm = \frac{2(M_\mathrm{d}^2+\Lambda^2-M^2)}{4M_\mathrm{d}^2-s}\left|1\pm\left\{1+\frac{\Lambda^2(s'-4M_\mathrm{d}^2)}{(M_\mathrm{d}^2+\Lambda^2-M^2)^2}\right\}^{\frac{1}{2}}\right|.$$

If $M_\mathrm{d}^2 < M^2+\Lambda^2$, the first term of (3.2) must be omitted, and one has a normal threshold case. The integration over α is trivially performed. The absorptive part of $F(s, \Lambda^2)$ in the anomalous region, $s_0 < s < 4M^2$, is

$$(3.5)\quad \mathrm{Im}\,F(s, \Lambda^2) = \Gamma_0^2\left[s(4M_\mathrm{d}^2-s)\right]^{-\frac{1}{2}}.$$

If this expression is considered in the non-relativistic limit, $s < 4M_\mathrm{d}^2$, then it becomes equal to the absorptive part calculated for a zero-range Schrödinger wave function, eq. (2.9).

It is clear that the anomalous region dominates the small momentum transfer behavior of the form factor and in addition, the imaginary part of F is particularly simple to calculate in this range of s. For later comparison $\mathrm{Im}\,F(s, \Lambda^2)$ is graphed in Fig. 2 for $\Lambda^2 = M^2$ (solid line). The relativistic expression for $\mathrm{Im}\,F$ is seen to be smaller than the Schrödinger expression on a zero-range wave function if $s > 4M^2$. Thus one might expect that the

relativistic corrections to the non-relativistic form factor are negative, and become important for momentum transfers of the order of the normal threshold, $4M^2$. This is in agreement with an earlier calculation based on the Bethe-Salpeter approach (6).

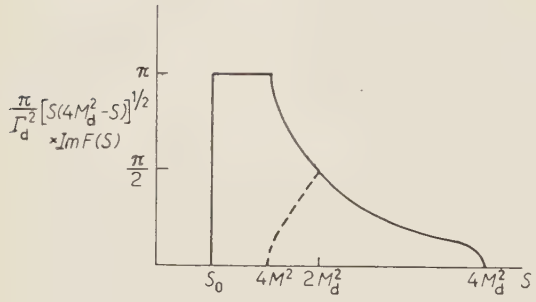

Fig. 2 (*).

There are several interesting comments one can make about Fig. 2. In the normal threshold case, $M_d^2 < 2M^2$, the graph does not pass through $\pi/2$ and simply vanishes at the normal threshold, $4M^2$. In both cases, it is clear that the absorptive part from this particular graph is an analytic function between physical thresholds. This property is probably a general result of perturbation theory (7), and may even have a more rigorous basis. The form factor will now be reconsidered from the dispersion theory point of view.

4. – Dispersion theory.

It is convenient for our purposes to discuss the form factor from the pair annihilation point of view. Therefore consider the vertex function

$$eF(s) = (4\bar{d}_0 d_0)^{\frac{1}{2}} \langle d\bar{d}^{(-)} | j^+(0) | 0 \rangle , \tag{4.1}$$

where j^+ is the scalar photon current operator. We now propose to proceed in the standard way and to contract the final deuteron. This is certainly correct in the physical region, $s > 4M_d^2$. The result is

$$eF(s) = i\sqrt{2\bar{d}_0} \int d^4x \exp[-id\cdot x] \langle \bar{d} | [J_d(x), j^+(0)] \theta(x_0) | 0 \rangle , \tag{4.2}$$

where an equal-time commutator contribution has been dropped for the moment. This form makes it plausible that in the rest frame of $\bar{d}$, $F(s)$ is an analytic function of $\bar{d}_0$ on the upper half plane. We will therefore make the reasonable assumption that $F(s)$ is analytic in the cut s plane, a fact which

(*) *Note added in proof.* – The dotted line in this graph should be multiplied by a minus sign. We thank Prof. OEHME for pointing out this error to us.

(6) R. BLANKENBECLER: reported in the *Annual International Conference on High-Energy Physics at CERN* (1958); edited by B. FERRETTI (Geneva, 1958).

(7) L. D. LANDAU: *Nucl. Phys.*, **13**, 181 (1959).

has been proven to all orders in perturbation theory if the masses satisfy certain inequalities (1). The absorptive part of $F(s)$ is conventionally written as

$$e \operatorname{Im} F(s) = \pi(2\bar{d}_0)^{\frac{1}{2}} \sum_t \langle \bar{d} | J_d(0) | t \rangle \langle t | j^+(0) | 0 \rangle \, \delta(d + \bar{d} - t) , \tag{4.3}$$

where t is a complete set of physical states. It is rigorously true that this statement of unitarity is applicable in the physical region. Our main concern will be the analytic continuation of this statement into the unphysical and anomalous regions. In order to compare with our previous discussion, the intermediate state consisting of a nucleon pair will be considered. In this approximation,

$$e \operatorname{Im} F(s) = \pi \int \frac{\mathrm{d}^3 n \, \mathrm{d}^3 \bar{n}}{(2\pi)^3} (2\bar{d}_0)^{\frac{1}{2}} \langle \bar{d} | J_d(0) | n\bar{n}^{(+)} \rangle \langle n^{(+)}\bar{n} | j^+(0) | 0 \rangle \cdot \cdot \delta(d + d - \mathcal{N} - \bar{\mathcal{N}}) . \tag{4.4}$$

Thus the deuteron form factor has been expressed as a folding of the nucleon form factor and the amplitude for the process $\mathcal{N} + \bar{\mathcal{N}} \to \mathrm{d} + \bar{\mathrm{d}}$.

The nucleon form factor is introduced as

$$(4\mathcal{N}_0 \bar{\mathcal{N}}_0)^{\frac{1}{2}} \langle \mathcal{N}\bar{\mathcal{N}}^{(+)} j^+(0) | 0 \rangle = e f^*_{\mathcal{N}}(s) , \tag{4.5}$$

where $f_{\mathcal{N}}(s)$ is analytic in the s-plane cut from $s = 4M^2$ to $s = \infty$. Actually, the threshold of $f_{\mathcal{N}}$ is determined by the particular model under discussion; for the physical deuteron it is $9\mu^2$, since the nucleon pair must be produced in a singlet isotopic spin state in order to couple to the deuteron pair final state. It will be assumed that $f_{\mathcal{N}}(s)$ is known from a study of the nucleon structure probelm.

Next the annihilation amplitude $(\mathcal{N} + \bar{\mathcal{N}} \to \mathrm{d} + \bar{\mathrm{d}})$ must be constructed. The invariant amplitude M is introduced as

$$M = (8\mathcal{N}_0 \mathcal{N}_0 \bar{d}_0)^{\frac{1}{2}} \langle \bar{d} | J_d(0) | \mathcal{N}\bar{\mathcal{N}}^{(+)} \rangle . \tag{4.7}$$

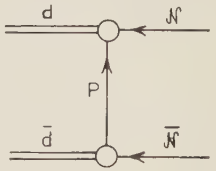

Fig. 3.

If the antinucleon is contracted, the contribution to M from the graph in Fig. 3 is easily isolated. This term, to be called the Born approximation, is found to be of the form

$$B = 8\Gamma_0^2 [(d - \mathcal{N})^2 + M^2)]^{-1} . \tag{4.8}$$

It is clear that what is needed for the evaluation of the absorptive part of the deuteron form factor is the S-wave projection of M in the center of

mass frame. If the following set of co-ordinates are introduced

$$\left\{\begin{array}{ll} N = (\mathcal{N}+\overline{\mathcal{N}}) = (d+\overline{d})\,, & N^2 = -s\,, \\ Q = \tfrac{1}{2}(\mathcal{N}-\overline{\mathcal{N}})\,, & 4|\boldsymbol{d}|^2 = s - 4M_d^2\,, \\ & 4|\boldsymbol{\mathcal{N}}|^2 = s - 4M^2\,, \end{array}\right. \tag{4.9}$$

then the S-wave projection of the Born term is

$$B_0(s) = \frac{8\Gamma_0^2}{\beta(s)} \ln\left[\frac{s - 2M_d^2 + \beta(s)}{s - 2M_d^2 - \beta(s)}\right], \tag{4.10}$$

where $\beta^2(s) = (s-4M^2)(s-4M_d^2)$.

It is convenient to express $B_0(s')$ in a form which makes transparent its analytical structure. This form for B_0 is

$$B_0(s) = 8\Gamma_0^2 \int_{-s_0}^{\infty} \frac{\mathrm{d}x}{(x+s)\beta(-x)} + 16\Gamma_0^2 \int_{-4M^2}^{-s_0} \frac{\mathrm{d}x}{(x+s)\beta(-x)}\,, \tag{4.11}$$

where $s_0 = 4M_d^2(1-(M_d^2/4M^2))$, and corresponds to the point where the exchanged nucleon is on the mass shell. This is an important point and will be returned to later. If $M_d^2 < 2M^2$, the second term in (4.11) does not appear.

Unitarity tells us that $M_0(s)$ has the phase of nucleon-antinucleon scattering in the S-state in the physical region. It is conventional to impose this restriction in the unphysical region also. We will assume that this is the correct procedure and later correct for the presence of the anomalous threshold. The standard approximate solution to this mapping problem is [8]

$$M_0(s) = B_0(s) + \frac{f(s)}{\pi} \int_{4M^2}^{\infty} \frac{\mathrm{d}s'\,|\boldsymbol{Q}'|\,B_0(s')}{(s'-s-i\varepsilon)\sqrt{s'}} \left[\frac{g(s')}{f(s')}\right], \tag{4.12}$$

where

$$g(s) = \frac{\sqrt{s'}}{|\boldsymbol{Q}'|} \exp[i\delta'] \sin\delta',$$

and

$$\ln f(s) = \frac{1}{\pi} \int_{4M^2}^{\infty} \mathrm{d}s'\, \delta(s')(s'-s-i\varepsilon)^{-1}.$$

[8] R. Omnès: *Nuovo Cimento*, 8, 316 (1958). g/f may also be written as $(\sqrt{s}/2i|\boldsymbol{Q}|)(1/f^* - 1/f)$.

The function $g(s)$ is the S-wave scattering amplitude which enjoys the standard analyticity properties. There is an important observation that must be made concerning the solution (4.12). The Born term has a branch cut along the negative real s axis up to $4M^2$, if $M_d^2 > 2M^2$. The dispersion line integral runs down to $4M^2$. Thus it is unclear whether or not this solution is on the physical sheet, since the line integral may cross the branch line of the integrand. In fact, it is not on the physical sheet, as will be made clear.

There are several equivalent ways of showing that M is not on the physical sheet. The first, and most direct, proceeds by making use of the fact that M is the boundary value of an analytic function in the mass and energy variables. Since the expression (4.12) for M is certainly correct if the external masses are sufficiently small, all that has to be done is to analytically continue to the physical mass values. This is the procedure developed by MANDELSTAM (9) who shows that as M_d increases, the line integral form $4M^2$ to ∞ must be deformed in order to avoid a branch line of $B_0(s)$ which depends on M_d.

There is another equivalent argument based perhaps more closely on physical requirements that yields the same result (*). It is clear that at threshold, the absorptive part of a scattering amplitude on the physical sheet must vanish because there is no phase space available to the particles in the intermediate state. This physical requirement is insured by the unitarity statement for the Green's function in momentum space. The fact that the phase shift δ goes to zero like $|\boldsymbol{Q}|$ at threshold, and the expression (4.11) for B_0 leads immediately to the conclusion that the absorptive part of M does not vanish at this point. This means that we are on the wrong Riemann sheet and must deform the straight line integral until a branch line of the integrand is crossed and the absorptive part vanishes as s approaches the point $4M^2$. These requirements lead uniquely to the path C illustrated in Fig. 4.

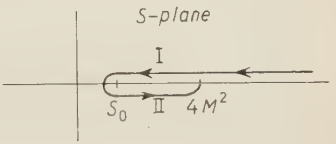

Fig. 4.

The analytic continuation is easily performed. In region I, we have

$$\int_{s_0}^{\infty} \frac{\mathrm{d}s' \, |\boldsymbol{Q}'| \, B_0(s'+i\varepsilon)}{(s'-s-i\varepsilon)\sqrt{s'}} \left[\frac{g(s'+i\varepsilon)}{f(s'+i\varepsilon)}\right].$$

In region II,

$$-\int_{s_0}^{4M^2} \frac{\mathrm{d}s' \, |\boldsymbol{Q}'| \, \tilde{B}_0(s'-i\varepsilon)}{(s'-s-i\varepsilon)\sqrt{s'}} \left[\frac{g(s'-i\varepsilon)}{f(s'-i\varepsilon)}\right],$$

(9) S. MANDELSTAM: *Phys. Rev. Lett.*, **4**, 84 (1960).

(*) This method of calculation was developed by the authors before the work of MANDELSTAM was known to us.

where $\widetilde{B}$ is the analytic continuation of B across the cut onto the second sheet

$$\widetilde{B}_0(s'-i\varepsilon) = B_0(s'-i\varepsilon) - \frac{16\pi\Gamma_0^2 i}{\beta(s')} . \tag{4.13}$$

Now g/f is real by construction ([10]), hence

$$\pi(M_0^c - B_0)/f = \int_{s_0}^{4M^2} \frac{\mathrm{d}s'\,|\boldsymbol{Q}'|}{(s'-s-i\varepsilon)\sqrt{s'}} \left[\frac{g(s')}{f(s')}\right] \left[B_0(s'+i\varepsilon) - B_0(s'-i\varepsilon) + \frac{2\pi i}{\beta(s')}\right] + \tag{4.14}$$

$$+ \int_{4M^2}^{\infty} \frac{\mathrm{d}s'\,|\boldsymbol{Q}'|\,B_0(s')}{(s'-s-i\varepsilon)\sqrt{s'}} \left[\frac{g(f')}{f(s')}\right].$$

Finally, $M_0^c(s)$ achieves the form

$$M_0^c(s) = B_0(s) + \frac{f(s)}{\pi} \int_{s_0}^{4M^2} \frac{\mathrm{d}s' \cdot 8\pi\Gamma_0^2}{(s'-s-i\varepsilon)\left[s'(4M_\mathrm{d}^2 - s')\right]^{\frac{1}{2}}} \frac{g(s')}{f(s')} + \tag{4.15}$$

$$+ \frac{f(s)}{\pi} \int_{4M^2}^{\infty} \frac{\mathrm{d}s'\,|\boldsymbol{Q}'|\,B_0(s')}{(s'-s-i\varepsilon)\sqrt{s'}} \frac{g(s')}{f(s')} .$$

The anomalous threshold in this scattering case can be looked upon as arising from an impulse approximation type graph in which the wave function for the bound nucleon provides the momentum behavior necessary to extend the cut from the normal threshold to s_0. It is also interesting that no off-energy shell ambiguities arise in this presentation of the impulse approximation. This will be discussed in more detail at a later time.

Turning back now to the deuteron form factor, its absorptive part in the physical region is easily evaluated according to eq. (4.4). The result is

$$\mathrm{Im}\, F(s) = \frac{1}{16\pi} \left(\frac{s-4M^2}{s}\right)^{\frac{1}{2}} f^*(s)\, M_0^c(s) . \tag{4.16}$$

In the unphysical region, if eq. (4.4) is evaluated in lowest order, *i.e.* $f(s) = 1$ and $M^c = B$, and the sum over states (which reduces to an angular integration) performed without worrying about analytically continuing the result from the physical region, the resulting absorptive part vanishes at threshold, $s = 4M^2$, and has a discontinuity in slope at $s = 2M_\mathrm{d}$. This is illustrated in Fig. 2 as

([10]) g/f will in general have left-hand cuts running into this region. But we may consider our problem as being an analytic continuation with respect to the mass Λ in B, so that we do not go round the branch points of g/f.

the dotted line below $2M_d$ and the solid line above. The requirement that this discontinuity lie below the normal threshold yields precisely the criteria found in perturbation theory ([1,2]) for the non-existence of anomalous thresholds. Actually, of course, the correct use of unitarity is to analytically continue the absorptive part from the physical region. If this is done, one is led onto the second branch of an arctangent at the point $s = 2M_d$. This is the solid curve of Fig. 2. This result then agrees with perturbation theory for $s > 4M^2$. All that remains is to include the anomalous region in the manner previously described.

The usual dispersion relation for the form factor is

$$F(s) = \frac{1}{\pi}\int_{4M^2}^{\infty} \frac{ds' \operatorname{Im} F(s')}{s' - s - i\varepsilon} . \tag{4.17}$$

The change of sheets is again accomplished by deforming the path of integration to C. In order to simplify comparison with our perturbation calculation, we will neglect the rescattering corrections to the form factor for the moment. In region I, $\operatorname{Im} s > 0$, we find from (4.16) and (4.11),

$$16\pi \operatorname{Im} F(s) = \left(\frac{s - 4M^2}{s}\right)^{\frac{1}{2}} B_0(s + i\varepsilon) , \tag{4.18}$$

where $(s - 4M^2)^{\frac{1}{2}} = i(4M^2 - s)^{\frac{1}{2}}$ if $s < 4M^2$. If $\operatorname{Im} s < 0$, then in region II the absorptive part is

$$16\pi \operatorname{Im} F(s) = i\left(\frac{4M^2 - s}{s}\right)^{\frac{1}{2}} \left[B_0(s - i\varepsilon) - \frac{16\pi i \Gamma_0^2}{\beta(s - i\varepsilon)}\right] . \tag{4.19}$$

Finally, if we write

$$F^c(s) = \frac{1}{\pi}\int_{s_0}^{\infty} \frac{ds' A(s')}{(s' - s - i\varepsilon)} , \tag{4.20}$$

the imaginary part of the form factor in the anomalous region is

$$A(s) = \Gamma_0^2 s \left[4M^2 - s\right]^{-\frac{1}{2}} , \tag{4.21}$$

and for s greater than $4M^2$ it is

$$A(s) = \frac{1}{16\pi}\left(\frac{s - 4M^2}{s}\right)^{\frac{1}{2}} B_0(s + i\varepsilon) . \tag{4.22}$$

This is identical to the result of perturbation theory. The anomalous threshold s_0 corresponds to the point where the exchanged nucleon is on the

mass shell. This is again the same as perturbation theory, where it is found that the anomalous threshold arises from all three particles being on the mass shell simultaneously. Thus the dispersion theory result is the same both numerically and physically as found in perturbation theory. The physical interpreation of the anomalous branch cut is found by comparison with the Schrödinger equation example of Section **2** ([11]).

The rescattering corrections can be easily included in the scheme in a similar way. Above the normal threshold, eq. (4.16) will remain unchanged. In the anomalous region, we have to take the difference of eq. (4.16) in I and II of Fig. 4. The Born and rescattering terms of eq. (4.15) yield respectively

$$\frac{\Gamma_0^2 f^*(s)}{[s(4M_d^2 - s)]^{\frac{1}{2}}} \qquad \text{and} \qquad -\Gamma_0^2 \left(\frac{4M^2 - s}{4M_d^2 - s}\right)^{\frac{1}{2}} \frac{f^* g}{s},$$

the sum of which becomes

$$\text{Im } F = \Gamma_0^2 \frac{1}{[s(4M_d^2 - s)]^{\frac{1}{2}}} f^*(s) \exp[2i\delta] = \Gamma_0^2 \frac{1}{[s(4M_d^2 - s)]^{\frac{1}{2}}} f(s) . \tag{4.23}$$

In order to perform the necessary analytic continuation, one must use the obvious relations $f^*(s) = f(s) S^{-1}$, where $S = \exp[2i\delta]$ is the S-matrix. This agrees with the Schrödinger picture according to which the deuteron form factor should be a folding of the nucleon form factor and the nucleon distribution.

One might hope that this procedure would also work for multiparticle intermediate states because the scattering amplitude should certainly have singularities when certain of the intermediate particles are on the mass shell. Since this is the source of the anomalous thresholds in perturbation theory, the continuation procedure described here should be applicable if the absorptive part has the required analyticity in the vicinity of the anomalous region.

5. – Discussion and conclusion.

It is interesting that one can carry out the previous dispersion theory discussion for what might be termed a covariant Hulthén deuteron model. This proceeds by evaluating the annihilation process not by dispersion theory but by some sort of Low-type equation. The difference is that there now appear vertex functions instead of the coupling constant Γ_0. That is

$$B(x) = 8\Gamma^2(x)(M^2 - x)^{-1} ,$$

([11]) See also ref. ([5]).

where $x = -(d-\mathcal{N})^2$, and $\Gamma(M^2) = \Gamma_0$ is the renormalized coupling constant. This vertex is given explicitly by

$$\Gamma(x) = (4d_0\mathcal{N}_0)^{\frac{1}{2}}\langle \mathcal{N}0\,|\,j_p(0)\,|\,d\rangle\,.$$

If this quantity is analyzed by taking the exchanged nucleon off the mass shell, it is found to satisfy a dispersion relation with an anomalous threshold present ([4]). This suggests an approximate evaluation by replacing the cut in Γ by a series of poles. It turns out to be slightly more convenient to approximate Γ^2 instead of Γ. If we set

$$\Gamma^2(x) = \Gamma_0^2\left|1 - \frac{2(M^2-x)}{M_1^2-x_1^2} + \frac{M^2-x}{M_2^2-x}\right|,$$

and calculate the absorptive part of $F(s)$ by the same methods as were used to arrive at (4.21), the result in the anomalous region is

$$A(s) = \Gamma_0^2\left[s(4M_d^2-s)\right]^{\frac{1}{2}}\left[\theta(s-s_0) - 2\theta(s-s_1) + \theta(s-s_2)\right],$$

where $s_{0,1,2}$ are the thresholds corresponding to the mass M^2, M_1^2, M_2^2. It is clear that the structure of the Hulthén model can be reproduced by suitably choosing the masses.

To recapitulate, the procedure suggested here for handling anomalous thresholds is as follows. First, calculate the absorptive part from unitarity in the physical region. Then continue this result to the normal threshold. If the absorptive part vanishes there, then this is a normal case, treatable by standard procedures. If, however, the absorptive part does not vanish, then one is faced with an anomalous threshold and the absorptive part in the anomalous region, as well as the correct threshold, is found by analytic continuation in the manner previously described.

* * *

The authors thank Brookhaven National Laboratory for its hospitality during the summer of 1959, when the main part of this work was done.

RIASSUNTO (*)

Si studia il fattore di forma di una particella nel cosidetto caso anomalo (sistema debolmente legato) dal punto di vista di: 1) la descrizione in termini di una funzione d'onda del tipo di Schrödinger; 2) la teoria della perturbazione e 3) la teoria della dispersione. Si da una prescrizione sul modo di calcolare la parte assorbitiva di una relazione di dispersione nel piano riemanniano appropriato.

(*) *Traduzione a cura della Redazione.*

Statistical Theory of Multiple Particle Production with Angular Momentum Conservation (*).

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(ricevuto il 3 Agosto 1960)

1. – In this note I should like to discuss a new formulation of Fermi's statistical theory (1) of multiple particle production, in which the condition of the angular momentum conservation is fulfilled and which can thus be applied to the analysis of peripheral collisions and angular distribution of secondary particles (*,*). It is hoped that such an approach will be useful for a more unified treatment of head-on and long-distance collisions. Of course it would be too optimistic if we expect that such a statistical consideration is sufficient for getting agreement with experiments, but in any case it will be reasonable to separate those « kinematical » effects which can be obtained without entering into detailed mechanism of the collision.

2. – The conventional formula for the relative magnitude of the phase space volume of n particles with total energy W and total linear momentum $\boldsymbol{P}$ is given by (*,*)

$$S_n(W;\boldsymbol{P}) = \frac{\mathrm{d}U_n}{\mathrm{d}W}, \tag{1}$$

$$U_n = \frac{\Omega^n \mathscr{L}^n}{(2\pi)^{3n} n!} \prod_{i=1}^{n} \left(\int \mathrm{d}^3 \boldsymbol{p}_i \right) \delta\left(\boldsymbol{P} - \sum \boldsymbol{p}_i\right) \Theta\left(W - \sum w_i\right), \tag{2}$$

(*) A fuller account of this work has been contributed to the *Acta Phys. Polon.*

(**) On leave from the Research Institute for Fundamental Physics, Kyoto University.

(*,*) Fermi treated this problem in his second paper (2) by a classical thermodynamical approximation; but the latter can be used only for a very large number of particles.

(*,*) $\hbar = c = 1$.

(1) E. FERMI: *Progr. Theor. Phys.*, **5**, 570 (1950).

(2) E. FERMI: *Phys. Rev.*, **81**, 683 (1951).

where

$$\Omega = \frac{4\pi}{3} R^3 : \text{ effective interaction volume}, \tag{3}$$

$$w_i = \sqrt{\boldsymbol{p}_i^2 + \mu^2}, \tag{4}$$

$$\Theta(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases} \tag{5}$$

$\mathscr{L}$ represents the effect of relativistic reduction of the interaction, which is usually visualized as the Lorentz contraction of the effective volume. Here we treat, for simplicity, the case of one kind of secondary bosons with no internal degrees of freedom (neutral pions, for example).

First we notice that (2) can be written as

$$U_n(W; \boldsymbol{P}) = \frac{1}{(2\pi)^{3n}\, n!} \prod_{i=1}^{n} \left| \int_{\Omega} \mathrm{d}^3 \boldsymbol{x}_i \int \mathrm{d}^3 \boldsymbol{p}_i \cdot \mathscr{L} \right| \cdot \tag{6}$$

$$\cdot \left| \prod_{i=1}^{r} \exp\left[i \boldsymbol{p}_i \boldsymbol{x}_i \right] \right|^2 \delta\Big(\boldsymbol{P} - \sum \boldsymbol{p}\Big)\, \Theta\Big(W - \sum w_i\Big),$$

which explicitly describes the original picture of Fermi: the square of the transition matrix element is regarded as being proportional to the probability of finding all the secondary particles, each of which is represented by a plane wave normalized in a big unit volume, simultaneously in the small effective volume of interaction.

3. – In (6) the final state of the system of created particles is specified in the representation in which each of them is labelled by its *linear momentum* $\boldsymbol{p}$ and the *total linear momentum* $\boldsymbol{P}$ is a good quantum number. Now we can use, however, an alternative representation, in which each particle is labelled by its *angular momentum* l and one of its components m, together with the magnitude of linear momentum p; and further, by taking a suitable linear combination of them, we can construct eigenstates of *total angular momentum* L and one of its components M.

In this way we are led to the following expression for evaluating the relative magnitude of the phase space volume of n secondary particles when the total energy W and the total angular momentum L, as well as one of its components M, of the final state of this assembly are specified (*):

$$S_n(W; L, M) = \frac{\mathrm{d} U_n(W; L, M)}{\mathrm{d} W}, \tag{7}$$

$$U_n(W; L, M) = \left(\frac{2}{\pi}\right)^n \frac{\mathscr{L}^n}{n!} \sum_{l_1, l_2, \ldots l_n; \lambda_1, \lambda_2, \ldots \lambda_{n-2}}^{(L)} \prod_{i=1}^{n} \left(\int_0^R \mathrm{d} r_i \int \mathrm{d}\omega_i \int p_i^2 \, \mathrm{d} p_i \right) \cdot \tag{8}$$

$$\cdot \Big| \sum_{m_1, m_2, \ldots m_n} K_i(L, M; l_1, \lambda_1, \ldots l_{n-2}, \lambda_{n-2}, l_{n-1}, l_n \,|\, l_1, m_1, \ldots l_n, m_n) \cdot$$

$$\cdot Y_{l_1 m_1}(\omega_1) \ldots Y_{l_n m_n}(\omega_n) \cdot j_{l_1}(p_1 r_1) \ldots j_{l_n}(p_n r_n) \Big|^2 \Theta\Big(W - \sum w_i\Big),$$

(*) Here I introduce the energy W and angular momentum L, M as independent parameters, by which a collision is to be specified. The object of this approach is to find out, assuming certain values of these parameters, what are the probable consequences, and not to discuss how these specified values have been realized.

In the above expression the first summation $\sum^{(L)}_{l_1, \ldots, l_n; \lambda_1, \ldots, \lambda_{n-2}}$ means that all possible combinations of integer values for $l_1, l_2, \ldots, l_n; \lambda_1, \ldots, \lambda_{n-2}$, which fulfil the following triangle relations should be taken (*):

$$
\left\{
\begin{array}{l}
|l_1 - \lambda_1| \leqslant L \leqslant l_1 + \lambda_1 , \\
\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
|l_{j+1} - \lambda_{j+1}| \leqslant \lambda_j \leqslant l_{j+1} + \lambda_{j+1}, \qquad (j = 1, 2, \ldots n-3), \\
\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\
|l_{n-1} - l_n| \leqslant \lambda_{n-2} \leqslant l_{n-1} + l_n .
\end{array}
\right. \tag{9}
$$

For a fixed set of $l_1, \ldots, l_n; \lambda_1, \ldots, \lambda_{n-2}$, the second summation goes over all values of m_i's ($-l_i \leqslant m_i \leqslant l_i$, $i = 1, 2, \ldots, n$). The coefficients

$$
K_n(L, M; l_1, \lambda_1, \ldots \lambda_{n-2}, l_{n-1}, l_n | l_1, m_1, \ldots, l_n, m_n) ,
$$

are the matrix elements of the unitary transformation from the representation, in which $l_1, m_1, \ldots, l_n$, and m_n are diagonal, to the one, in which $l_1, \ldots, l_n; \lambda_1, \ldots, \lambda_{n-1}$, as well as L and M are diagonal. They can be constructed by repeated application of the addition of two angular momenta and can easily be expressed in terms of the Clebsch-Gordan coefficients. It is seen that they have the following properties:

$$
K_n(L, M; l_1, \lambda_1, \ldots, \lambda_{n-2}, l_{n-1}, l_n | l_1, m_1, \ldots, l_n, m_n) = 0 , \tag{10}
$$

if $m_1 + m_2 + \ldots + m_n \neq 0$.

$$
\sum_{m_1, m_2, \ldots m_n} |K_n(L, M; l_1, \lambda_1, \ldots, \lambda_{n-2}, l_{n-1}, l_n | l_1, m_1, \ldots, l_n, m_n)|^2 = 1 . \tag{11}
$$

We can perform angular integration in (8) with the help of (11) and the orthonormal relation of Y_{lm}'s, and obtain

$$
U_n(W; L, M) = \left(\frac{2\mathscr{L}}{\pi}\right)^n \sum^{(L)}_{l_1, \ldots, l_n; \lambda_1, \ldots, \lambda_{n-2}} \prod_{i=1}^{n} \left[\int_0^R \mathrm{d}r_i \int_0^\infty \mathrm{d}p_i \{p_i r_i j_l(p_i r_i)\}^2 \right] \cdot \Theta\Big(W - \sum w_i\Big) . \tag{12}
$$

Here the factor $\{prj_l(pr)\}^2$ is appreciably different from zero only when $pr \gtrsim l$, and restricts the allowed region of p or l. In other words, our interaction volume is not uniformly effective for the partial wave of given angular momentum owing to the presence of centrifugal barriers.

From (7) and (12) we can work out the probable multiplicity distribution of

(*) In practical calculations we have to take into account the degeneracy due to the indistinguishability of particles. Further, we can replace the summation over λ's by suitable coefficients.

secondary particles when the total energy and the total angular momentum are specified (*). It depends on L but not on M, as can be expected.

4. – The angular distribution of secondary particles, which is certainly a function of L and M, is obtained by leaving one angular integration unperformed when we go over from (8) to (12) and thus expressing S_n in the following way:

$$S_n(W; L, M) = \int \mathrm{d}\omega \sum_{l,m,m'} Y^*_{lm}(\omega)\, Y_{lm'}(\omega) F^{(n)}_{l,m,m'}, \tag{13}$$

with

$$F^{(n)}_{l,m,m'} = \sum_{\lambda}^{(L)} C^*(l, \lambda, L; m, M-m, M)\, C(l, \lambda, L; m', M-m', M) \cdot$$

$$\cdot \int_0^R \mathrm{d}r \int_0^\infty \mathrm{d}p \cdot \mathscr{L} \cdot \{pr j_l(pr)\}^2 S_{n-1}(W-w; \lambda, \mu)\,. \tag{14}$$

The summation $\sum_{\lambda}^{(L)}$ is taken over all values of λ which satisfy the triangle relation with L and l. The integrand in (13), when averaged over all possible n's, gives the expected angular distribution.

5. – The following tables show a numerical example for the case (**) of $W = 10\ \mu$m, $R = 1/\mu$, $\mathscr{L} = 0.39$, and various possible values of L. These results have been obtained by very crude approximations, for instance replacing the factor $\{pr j_l(pr)\}^2$ by a step function (*,*) $\Theta(pr - l)$, just to see some general aspects of our approach, and so the numbers cannot be taken seriously. More realistic applications, which could be compared with experiments, are now being investigated.

TABLE I. – *Multiplicity distribution of the secondary particles.*
($W = 10\ \mu$, $R = 1/\mu$, $\mathscr{L} = 0.39$)

n \ L	0	1	2	3	4	5	6	7	8	9
1	18%	10%	10%	11%	14%	19%	29%	47%	67%	100%
2	44%	46%	48%	51%	54%	57%	58%	49%	33%	0%
3	29%	34%	34%	32%	28%	22%	14%	4%	0%	0%
4	7%	9%	8%	6%	4%	2%	0%	0%	0%	0%
$\geqslant 5$	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
$\overline{n}$	2.3	2.4	2.4	2.3	2.2	2.1	1.8	1.6	1.3	1.0

(*) See footnote on p. 609.

(**) If we take the Lorentz contraction literally, this case corresponds to a nucleon-nucleon collision of incident energy (in the laboratory system) ~ 11 GeV, and inelasticity ~ 0.5 (cfr. the footnote on p. 610).

(*,*) Θ is defined by (5).

TABLE II. – *Distribution of angular momenta of the secondary particles* (*).
($W=10\ \mu$, $R=1/\mu$, $\mathscr{L}=0.39$)

l \ r	0	1	2	3	4	5	6	7	8	9
0	55%	30%	25%	21%	18%	16%	13%	11%	7%	—
1	32%	48%	34%	28%	22%	18%	14%	8%	7%	0%
2	11%	17%	31%	23%	19%	16%	12%	9%	5%	0%
3	2%	4%	8%	23%	16%	14%	11%	7%	5%	0%
4	0%	0%	1%	5%	21%	13%	10%	8%	5%	0%
5	—	0%	0%	1%	3%	23%	11%	8%	6%	0%
6	—	—	—	0%	0%	1%	27%	7%	5%	0%
7	—	—	—	—	—	0%	1%	42%	7%	0%
8	—	—	—	—	—	—	—	0%	57%	0%
9	—	—	—	—	—	—	—	—	—	100%
$\bar{l}$	0.6	0.9	1.2	1.7	2.1	2.7	3.4	4.6	6.1	9.0

(*) This gives a rough idea of the angular distribution.

* * *

The author expresses his gratitude to the Polish Academy of Sciences and Professor L. INFELD for the hospitality.

Note added in proof.

After sending this article for publication I noticed that the same idea as presented here had been once developed by COOK in connection with the analysis of the nucleon-antinucleon annihilation. (See LE ROY, FRANKLIN and COOK: *Thesis*, UCRL-8841, 1959).

IL NUOVO CIMENTO Vol. XVIII, N. 3 1° Novembre 1960

Field Metrics.

II. A Correction.

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(ricevuto il 5 Settembre 1960)

In a previous paper [1] it has been suggested that the field metrics and commutators are correlated but the eq. I. (8),

$$T_B(\varkappa\tau(\bar{1}))\, T_B^*(\varkappa\tau(\bar{2})) = \tau(\bar{1}\,\bar{2})\, J_1(\varkappa\tau(\bar{1}\,\bar{2})), \tag{1}$$

does not have a solution as is readily seen by putting $\tau(\bar{1})$ or, and $\tau(\bar{2})$ equal to zero.

To maintain the above hypothesis it is therefore necessary to postulate further that the τ space is also four-dimensional,

$$\tau(\bar{1})^2 = \tau_0(\bar{1})^2 - \tau_1(\bar{1})^2 - \tau_2(\bar{1})^2 - \tau_3(\bar{1})^2, \tag{2}$$

and then

$$(x(\bar{1}\,\bar{2})^2 - \tau(\bar{1}\,\bar{2})^2)\, \pi(\bar{1}), \pi(\bar{2}) = 0 \sim [\pi(\bar{1}), \pi(\bar{2})] = \frac{\varepsilon(x_0(\bar{1}\,\bar{2}))\,\varepsilon(\tau_0(\bar{1}\,\bar{2}))}{\tau(\bar{1}\,\bar{2})^2}\, \delta(x(\bar{1}\,\bar{2})^2 - \tau(\bar{1}\,\bar{2})^2), \tag{3}$$

$$\frac{\varepsilon(x_0)\,\varepsilon(\tau_0)}{\tau^2}\, \delta(x^2 - \tau^2) = \frac{1}{(2\pi)^4} \iint \exp\,[ip\cdot x + i\varkappa\cdot\tau]\, \frac{\varepsilon(p_0)\,\varepsilon(\varkappa_0)}{\varkappa^2}\, \delta(p^2 - \varkappa^2)\, \mathrm{d}^4p\, \mathrm{d}^4\varkappa. \tag{4}$$

The conservation of energy and momentum is obtained by interpreting the $\varkappa_\mu$ co-ordinates as internal energy and momentum co-ordinates and we conclude that the existence of a metric or the commutation relations of the form (3) indicate a non local field theory. Furthermore now field interactions (I Sect. 3) of vector and pseudovector types are favoured.

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[1] P. Sen: *Nuovo Cimento*, **15**, 513 (1960), which is called I here.

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Recensioni.

BENO GUTEMBERG – *Phisics of the Earth's Interior* – Int. Geophysics Series – Vol. 1. – Academic Press – New York and London – 1959 (pp. 240, numerose figure).

È l'ultima opera del grande geofisico scomparso il 25 Gennaio scorso. Sintetizza i risultati compiuti nell'ultimo cinquantennio nell'ambito della Geofisica, con particolare riguardo alla Sismologia.

Inizia con una critica sui metodi di ricerca in Geofisica e sulla accuratezza dei risultati. Un esame particolare è riservato alla crosta terrestre. Dopo una rapida esposizione dei metodi atti alla determinazione delle velocità di propagazione delle onde elastiche negli strati e del loro spessore, vengono riassunti i risultati ottenuti nelle più recenti, accreditate ricerche. Un capitolo è riservato al mantello terrestre (che va dalla superficie di Mohorovicic — 35 km di profondità media — al nucleo, alla profondità di 200 km circa). Si accenna alla astenosfera, alla zona di rapido aumento delle velocità fra 200 e 950 km e alla regione stendentesi fra questa profondità e il nucleo. Particolare attenzione è riservata al nucleo: ai tempi di tragitto attraverso di esso, alla parte fluida del nucleo, al nucleo interno (probabilmente solido), alla sua composizione e alle probabili relazioni fra la parte fluida del nucleo e il campo magnetico terrestre.

Viene esposto il problema della temperatura e dei processi termici nell'interno della Terra (temperatura e gradiente termico alla superficie della Terra, nella crosta, nelle zolle continentali, al fondo degli oceani, conduttività termica nella Terra, correnti di calore; generazione del calore nella Terra, ecc.).

Altri capitoli sono dedicati alla densità, pressione e gravità nella Terra; alle costanti elastiche e ai processi elastici (costanti elastiche, mutazione, variazioni della latitudine, maree terrestri libere vibrazioni della Terra, rigidità compressibilità, modulo di Young, ecc.); nonchè ai processi non elastici (attenuazione delle onde elastiche nella Terra, viscosità negli strati esterni, correnti convettive, movimenti relativi dei grandi blocchi della crosta, movimenti secolari dei poli, ecc.).

In complesso, l'opera costituisce una preziosa messa a punto sulle attuali cognizioni nel campo della Geofisica e sui problemi che ancora attendono una soluzione.

Un'adeguata attenzione è riservata al contributo di studiosi italiani.

P. CALOI

W. I. BARANOW – *Radiometrie*. B. G. Teubner Verlagsgesellschaften, Leipzig, 1960; pagg. VII+422; DM. 33.

Questo libro è la seconda edizione, tradotta in lingua tedesca, di un'opera russa, che tratta dei fondamenti scientifici della radioattività e dei metodi di misura delle radiazioni, con particolare riguardo alla metodologia dell'analisi

radiometrica dei campioni di rocce, alla prospezione radiometrica e in generale all'impiego dei metodi della fisica nucleare alla risoluzione di problemi di carattere geologico.

Come l'Autore riferisce, nello sviluppo dei metodi radiometrici gli scienziati sovietici hanno avuto un ruolo importante. All'accademico W. I. Wernadski si attribuisce il merito di aver introdotto la radiogeologia nel novero delle discipline geologiche, il geofisico L. N. Bogojawlenski per primo impiegò il metodo radiometrico per individuare la distribuzione degli elementi radioattivi in determinate configurazioni geologiche, la definizione degli orizzonti stratigrafici mediante misure di attività γ fu proposta dai geofisici sovietici G. W. Gorschkow e L. M. Kurbatow.

Il materiale raccolto e ordinato in questo volume è notevole, e in questo senso l'opera può essere considerata un manuale di consultazione e di orientamento su questioni specifiche veramente prezioso. Il libro è suddiviso in sette capitoli. I primi tre sono di carattere generale, e raccolgono tutte quelle nozioni sulla radioattività e sui metodi di rivelazione che sono considerate sufficiente presupposto agli argomenti specifici che seguono. Nei tre capitoli successivi è posto in evidenza il significato della dinamica delle reazioni nucleari in Geologia, e sono descritti metodi e strumenti dell'analisi e dell'indagine radiometrica. L'ultimo capitolo riguarda gli effetti biologici delle radiazioni.

Se i procedimenti adottati per l'enunciazione delle proprietà fondamentali della radioattività possono essere discutibili da un punto di vista critico, peraltro apprezzabile appare la parte propriamente sperimentale: ogni metodo è illustrato da esempi applicativi, fra i quali alcuni appaiono particolarmente notevoli ed interessanti. Chiarezza di esposizione, ricchezza di dati numerici selezionati, di grafici, schemi e fotografie, fanno si che pur nella inevitabile provvisorietà di molti articoli, il volume costituisca un utilissimo strumento di informazione e di lavoro, che consigliamo a tutti coloro che s'occupano attivamente dell'argomento.

C. MANDUCHI

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